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Fixed income management: an active strategy to optimize investments timing.

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Abstract

The goal of this work is to apply an active management strategy to a financial portfolio composed by fixed income securities. We use Vasicek and Cox Ingersoll Ross (CIR) interest rate models to exploit projections of future interest rate levels. The first part is dedicated to a brief overview of the fixed income market especially to understand the different measures available to compute the array of risks that an investor has to consider when investing in fixed income securities. The second part provides a comparison between the characteristics of Vasicek and CIR models. They are the so called "short interest rate models" which are used to describe the future evolution of interest rates. The analysis focuses on the most important characteristics of these models: mean reversion effect and volatility. Finally, we consider an optimization model with the aim of improving the total return on the initial wealth exploiting the interest rate levels and the investment timing. The strategy consists in shifting the maturity composition of the portfolio according to the expectation given by the models. Comparisons with other investment strategies are provided.

Introduction

The reason that drives active bond portfolio management is the confidence about the ability of managers to correctly timing the variations of interest rates. Fixed income market is a complex system. An asset manager must have good knowledge in various fields (economics, politics, history, psychology, statistics). First, it is essential to have a deep understanding of the assumptions underlying the forecasts and the characteristics of the market where these predictions are going to be applied. Second, the manager has to compare the predictions with "forward" market prices and look for investment opportunities if his predictions deviate from market expectations. The confidence in the forecasts is one of the most important judgments of the asset manager. The qualitative predictions can affect the quantitative ones. For instance, high confidence on a bullet strategy with a lack of confidence on the forecasts of yield curve steeping will lead to opt for a high duration portfolios, excluding the considerations on the curve. On the other hand, criticisms to this approach point the attention to the weakness of the active management first of all, the higher costs. Commissions charged on each operations make the active strategy more expensive than a passive strategy. The goal of this work is to apply a Mean Absolute Downside Deviation model (MADD) on a portfolio composed by three Sovereign bonds issued by the Italian Government with maturities of 10, 3 and 1 year. The optimization procedure is based on rates projections of short term interest rate models calibrated with a least squared regression method. Moreover, we try to overcome the weak point of the short rate equilibrium models that is the non perfect fit to the actual term structure connecting the forecast variations with the real observed data. A specific benchmark is not available in the real market, so we provide an equally weighted portfolio composed by the same three bonds. The results demonstrate that by moving the quotes invested in each bond (and in turn the portfolio's duration) the optimized portfolio is able to obtain a better performances. In the first chapter we give an overview on the characteristics of fixed income securities and active management strategies. The second chapter is dedicated to the description of short rate models and their calibration. The third chapter recalls few optimization models and the issues about benchmark selection criteria. In the fourth and final chapter will present the data set, the estimates and the results of the empirical application.

Chapter 1 - Fixed income portfolio management. An overview.

1.1 Investing in fixed income markets.

Investment in fixed income assets has always been one of the most considered among the investment choices available for investors. Maybe because people who want to plan long term investments have a sort of "low risk" perception when they deal with those instruments. Investing in fixed income assets is very different from investing in any other asset class. First of all, there is a change in prospective, because the investor assumes the typical position of the "lender". Indeed, when the object of the trade is a bond, we have a situation where the debt obligations of the borrower (big firms or companies, banks, governments) has to be settled by paying back to the lender (investor) a certain amount of money. So the borrower has to pay the face value of the debt (principal) plus the interests, within a fixed period of time (maturity) that can vary from months to years. The legal agreement between the borrower and the lender is called bond "indenture". Fixed income markets can be divided in sectors[1]:

- Treasury, who includes three types of securities: Treasury bill, Treasury note and Treasury bond. Usually a Treasury bill matures within 1 year, while Treasury note goes up to 10 year maturity and, finally, the Treasury bonds have maturities over 10 years;
- Agency sector includes federally related institutions;
- Municipal, regarding the local government;
- Corporate. In this sector we have securities issued by corporations to finance their operations. They can emit bonds, medium term notes and commercial paper for very short periods. This sector can be divided in investment grade and non investment grade. This classification reflects the higher or lower probability of default of the borrower;
- Asset backed securities. The securitizations is a process of pulling assets together in order "to back" more security emissions;

[1] Frank J. Fabozzi, T. Dessa Fabozzi , "Bond markets, analysis and strategies", Englewwod Cliffs, N.J. Prentice-Hall International, Inc., pp.,80-97.

- Mortgage loans. They represents residential or commercial loans usually secured by a real estate. As the distinction in corporate sector the residential mortgages can be divided in prime and subprime based on the probability of default of the borrower.

In bond markets we have some key concepts like the "volatility". As in all financial fields the volatility mirrors the risk of the asset. We said that the risk perception of a bond investor is weaker then the risk perception of a stock investor. But there is a wide range of risks to take into consideration when we are investing in fixed income securities[2]:

- Interest rate risk. The price of a bond goes up whenever interest rate goes down. The inverse relationship is given by the formula of the price of a bond:

$$P = C\left(\frac{1-(1+i)^{-N}}{i}\right) + M\,(1+i)^{-N}$$

  Where:

  - "C" is the coupon payment;
  - "N" is the number of payments;
  - "M" is the face value;
  - "i" is the yield to maturity.

- Reinvestment risk. It is the risk of being able to reinvest the cash flow (coupons) coming from the investment at the same rate in each period.
- Call risk. It is related to the Reinvestment risk but it includes the risk of being able to reinvest the entire principal at maturity.
- Credit risk. It is also known as "default risk", when the lender is not able to pay back its obligation. It is evaluated by a credit rating who gives the probability of default of the borrower. They are computed by Credit Agencies. The increase of the return that compensates the higher credit risk is called credit spread.
- Downgrade risk. It is the risk related to credit rating declassification of a particular bond issuer.

[2] Frank J. Fabozzi, T. Dessa Fabozzi , "Bond markets, analysis and strategies", Englewwod Cliffs, N.J. : Prentice-Hall International, Inc., pp.,1-6.

- Inflation risk. It runs two ways: the rise of inflation, the interests rate increase pushing down the price of bonds (Fisher equation). Secondly, the Purchasing Power risk (PP risk) is the risk that the same amount of money will worth less than before the inflation effects appeared.
- Exchange rate risk. Variations in currency value can change the value of the bond.
- Liquidity risk. With low liquidity, the risk is to not be able to sell the bond to the market (frozen market). This results in a fall of the price of the bond.
- Volatility risk. The value of structured bonds can be influenced by volatility of their option components.
- Risk risk. This is the unknown risk that cannot be estimated. This is the same as uncertainty. The difference is that the risk is a probability that can be estimated while the uncertainty there is no measurable estimation models (Tsunami, Nuclear war,…).

For all the risks there is a compensations given by a "premium" component.

1.2 *Interest rates and bonds evaluation.*

The most famous are probably the *mortgage rates* that a bank asks to its clients when they want to raise money to buy a real property or in case they are already property owners to raise funds for any purpose; another type of interest rate is the *treasury rates* at which the Governments fund their economical and social acts. Leading banks use the *London Interbank Offer Rate* or *Libor* to charge the principal they borrowed each other. *Repurchase agreement rate* or *Repo* are used in trading activities and they correspond to the difference between forward rate negotiated in T_0 and the spot rate in T_1.

Recalling the formula above we know that the theoretical price of a bond corresponds to the actualization of its future cash flows:

$$P = C\left(\frac{1-(1+i)^{-N}}{i}\right) + M\,(1+i)^{-N}$$

The interest used to calculate this price is called *yield to maturity,* and it represents the *Internal Rate of Return* (IRR) of an investment. The interests relative to an investment with maturity of "n" years and without any intermediate coupon payments are called "*zero rates*" or "*zero coupon rates*". They can be observed looking at the *strip bonds* quotations or calculated by using the so called b*ootstrap method.*

1.3 Term structure theories.

The term structure of interest rates describes the relationship between interest rates bonds maturities. Sometimes it is also known as a yield curve and it is a very important reference point in economy. The shape of the term structure is given by the expectations of the investors about future changes in interest rates. Different expectations imply different shapes of the yield curve, and in order to create projection on its future movements, it is needful to look at the different theories underlying these changes. There are basically three theories about the shape of the term structure:

a. The **expectation theory** states that long term interest rates are determined by the expectations on the forward rate structure. This theory abstracts from some complications in the real world such as transaction costs and taxes so investors will drive forward rates to the level of future short-term rates.
b. The **market segmentation theory** says that the is not a necessary relationship between short, medium and long term rates. Interest rate levels are simply given by supply – demand pricing process.
c. The **liquidity preference theory** is based on the assumption that investors "prefer" liquidity so they tend to invest for short periods while companies and institutions "prefer" to borrow for longer period. Those behaviors lead to an upward shape of the yield curve where forward rates are higher than the expectations on future spot rates.

Four kinds of yield curve have been historically observed in the market. The most observed is the Normal Yield Curve composed by low rates on short maturities and high rates on medium and long maturities. The positive slope reflects expectations for the economy to grow in the future and, in turn, a rise in the inflation level. Investing on long maturity bonds typically require higher interest rate to compensate for the higher risk of default of the issuer. Furthermore, the capital invested for a long period cannot be used for others purposes; for this reason investors require a compensation for the time value of money.

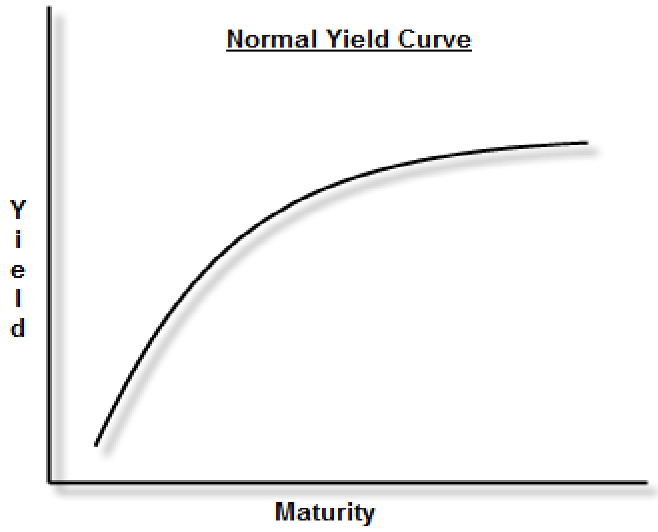

Figure 1. Normal yield curve composed by low rates on short maturities and high rates on medium and long maturities. It reflects expectations about future grow of the economy. (source: blog.knopman.com)

A negative sloping yield curve appears in the case of negative economic projections, where high interest rate in the short term mirrors a lack of confidence in the market and generates the yield curve inversion. This phenomenon called "flight to quality" drives investors to move their preferences towards safer investments (Figure 2).

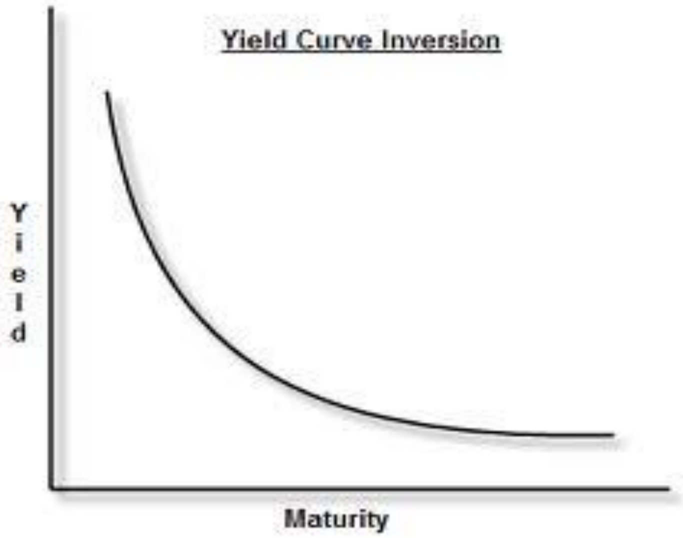

Figure 2. Negative slope of the yield curve. It reflects negative expectations about future economic developments. (source: www.mysmp.com)

A flat term structure describes a situation where for all maturities all interest rates values are the same or there is a small difference between short and long term interest rates. The flattering of the yield curve can be explained by a combinations of two effects: inflation expectations and monetary policy. The former tends to move upward or downward the long term part of the curve while the latter mainly acts on the short term part. Indeed, an expansionary monetary policy has the purpose to increase the total supply of money by lowering the interest rates; on the opposite, a contractionary monetary policy acts to slow inflation by increasing the interest rate levels. A flattering of the yield curve can be driven by:

a. The combination of lower inflation expectation and contractionary monetary policy, when the slope of the term structure before the shock was positive or normal;
b. The combination of higher inflation expectation and expansionary monetary policy, when the slope of the term structure before the shock was negative or inverted;

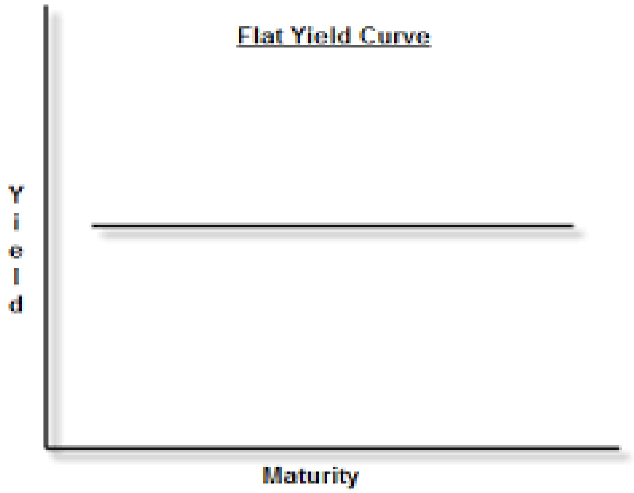

Figure 3. Flat Yield Curve. It reflects a mix of positive and negative expectations about inflation levels and monetary policies. (source: blog.knopman.com)

1.4 *How to measure the risk of a bond.*

As we have seen before, the equation of the bond price provides and inverse relationship between price and risk. But each bond has its own "risk profile" given by the variation of three elements[3]: yield to maturity; credit rating and maturity.

[3] Fabozzi, Frank J. 1997 "Managing Fixed Income Portfolios", Hand book Frank J. Fabozzi Associates, pp.1-17.

The effect of yield to maturities on bond price depends upon the magnitude of its changes. For little variations of the yield to maturity the effect is symmetric. This means that positive and negative change in the price of the bond are equal. For example, let's consider a bond with face value of 100 Euros, coupon of 5% and YTM of 2,5%. As shown in Table 1, when interest rates rise/fall by 5 basis points the price of the bond will rise/fall by 0,15 Euros. If interest rates face a bigger variation (50 basis points), the price variation will not be proportional: when interest rates rise to 3% the change in price "delta" is equal to 1,483, while in case of negative shock with interest rates at 2%, delta is equal to 1,512. Thus, the effect on large variation of the yield is asymmetric with bigger absolute values for the negative shocks.

Face Value	Maturity	Coupon	Ytm	Price	Δ	
100	3	5%	2,50%	107,1401	-	
100	3	5%	2,55%	106,9905	0,150	ΔYtm = 5 b.p.
100	3	5%	2,45%	107,2899	0,150	
100	3	5%	3,00%	105,6572	1,483	ΔYtm= 50 b.p.
100	3	5%	2,00%	108,6516	1,512	

Table 1. Symmetric and asymmetric variations on bond prices.

The positive relationship risk-maturities simply derives from the price formulation of the bond. The longer the maturity, the higher the number of payments that have to be actualized. So a raise of the left side of the equation simply force the right side to raise up as well.

The negative relationship between credit rating and risk is closely related to the default risk of the lender. The default premium is one of the components of the nominal interest rate. As we said before the interest rate is a sum of risk free rate plus different risk premium who compensate for widen risk[4]:

$$i_n = i_{RF} + i_{PP} + i_l + i_d$$

Where:

- i_n: Nominal interest rate;
- i_{RF}: Risk Free interest rate;

[4] Frank J. Fabozzi, T. Dessa Fabozzi , "Bond markets, analysis and strategies", Englewwod Cliffs, N.J. : Prentice-Hall International, Inc., pp.,15-20.

- i_{PP}: Purchasing Power risk (inflation);
- i_l: Liquidity risk;
- i_d : Default or credit risk.

So the raise of default premium caused by a downgrade of a lender, should widen the interest rate levels. Even though we are in a downgrade contest, sometimes there is a mix of effects that can instead push interest rates down: the downgrade of U.S. Treasury bonds from AAA to AA+ in 2011 has not been followed by a raise in the interest rates who remained very low. This is because the risk free component falls at the same time. As the economy moves into recession, suddenly the free risk interest rates fall significantly as they mirror a weaker economy characterized by less credit granted by banks or institutions.

The most common ways to measure the volatility of a bond are the Present Value of a Basis Point (PVBP) and the Duration. The former measures the bond price variation due to an oscillation of 1 basis point of the yield to maturity usually expressed as absolute value variation. As we saw in Table 1, the bigger the variation, the bigger the volatility. When we deal with different series of variations the final PVBP is a simple average of PVBP of the single variations. Duration is defined as "a measure of the average life of a security"[5]. Also called Macaulay duration it represents a weighted average of dates of cash flows guaranteed by a bond where the weights are the present values of each cash flow divided by the price of the bond:

$$D = \sum_{i=1}^{N} \frac{C_i \, (1 + yield)^{-t_i}}{P} * \, t_i$$

Where:

- $i = 1, \ldots .N$ is the number of cash flows;
- C_i: the amount of each cash flow;
- P: price of the bond;
- t_i: date of cash flow payment.

[5] Gerald O. Bierwag, 1987, "Duration analysis, Managing the interest rate risk", Journal of Banking and Finance,12, pp.161-166.

The Duration has a particular relationship with the fluctuation (volatility) of the price that can be defined as:

$$\Delta\%Price = -\frac{1}{(1+yield)} * D * Yield\ variation * 100$$

The precision of this formula is not really high. Indeed, the approximation of the price variation is good only for little changes in the yield. If we consider bigger variations, the result will just approximate. Another measure is the so-called "Modified Duration" who is easy and fast to be compute because it simply corresponds to the ration between the Macaulay Duration and one plus the yield:

$$Modified\ Duration = \frac{D}{(1+yield)}$$

The estimation error in computing the price using Duration method can be represented in a graph where the distance between the actual price curve and the duration line indicates the approximation of the real price value (Figure 4). In order to obtain a more precise value we should take into consideration the "convexity" that is a mathematical approach to calculate the value of this distance. The concept is that by looking to major order derivatives, the estimation becomes more and more precise. Thus the convexity is the part of the variation that we cannot compute by using the Macaulay Duration. The formula is:

$$\Delta\%Price = -MD * Yield\ variation * 0{,}5 * Convexity * (Yield\ variation)^2$$

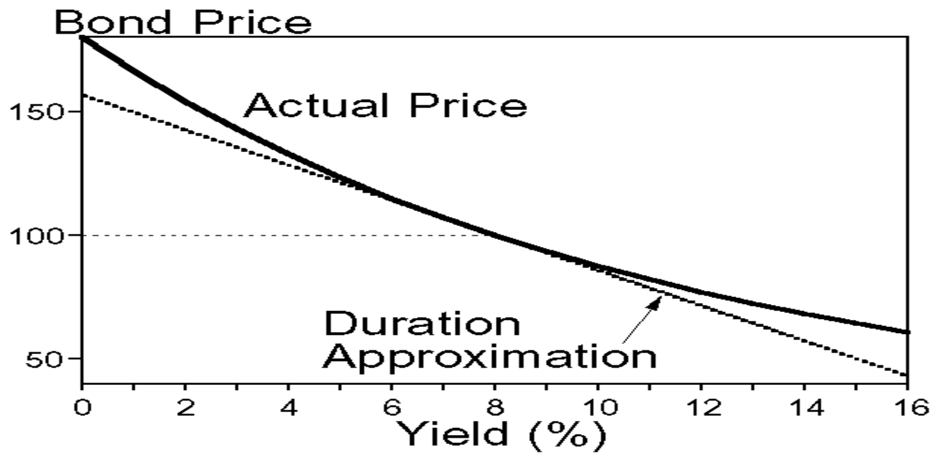

Figure 4. Duration approximation represented by the straight line is a good measure for small price variations while for larger variations the approximation is not fitting the actual

price curve. The gap can be eliminated by including the convexity on the price variation formula (source: merage.uci.edu)

Another way to calculate the duration is to consider the approximate percentage change in the price for a specific variation of basis points[6]:

$$Approximate\ duration = \frac{P_- - P_+}{2P_0(\Delta yield)}$$

Where:

- P_0 is the price of the bond;
- P_+ is the new price after a positive shock;
- P_- is the new price after a negative shock.

By applying the same principle we can also compute the approximate convexity as[7]:

$$Approximate\ convexity = \frac{P_- - P_+ - 2P_0}{2P_0(\Delta yield)^2}$$

When the analysis focuses on portfolios of bonds, the way to calculate the duration is simply the weighted average of the duration of the bonds in the portfolio where w_i are the weights of each bond:

$$Portfolio\ Duration = w_1 D_1 + w_2 D_2 + \cdots + w_N D_N$$

We present an example in which we consider a portfolio composed by 3 kind of bonds A,B,C with 3 different durations, we can calculate the portfolio duration using the formula above.

In Table 2, the market value of each bond is used to compute the weight of each bond in the portfolio. Bond C represents the biggest percentage of portfolio composition but it’s duration of 10 years is not representative of the entire portfolio because it also includes 6,5 years duration of Bond A and 4,3 years of Bond B.

[6] Ho, Thomas S.Y., 1992, "Key Rate Durations: Measures of Interest Rate Risks". *Journal of Fixed Income* 2: 29–44.

[7] Fabozzi, Frank J. (1999), John Wiley and Sons, "The basics of duration and convexity", *Duration, Convexity, and Other Bond Risk Measures*, Frank J. Fabozzi Series.

Title	Marketvalue	Duration	Weight	D*W
Bond A	2.000.000	6,5	0,28	1,82
Bond B	500.000	4,3	0,07	0,301
Bond C	4.800.000	10	0,65	6,5
Tot.	**7.300.000**	-	1	8,621

Table 2. Duration of bond portfolio example.

The portfolio's duration is equal to 8,621. This means that when the interest rate has a positive shock of 100 basis points, then the value of the portfolio will change by approximately 8,621%. This mechanism works only if each bond in the portfolio changes by the same amount of basis points, so that "Δ yield" is the same for all the bonds. The only way this can happen is when there is a "parallel shift" of the yield curve.

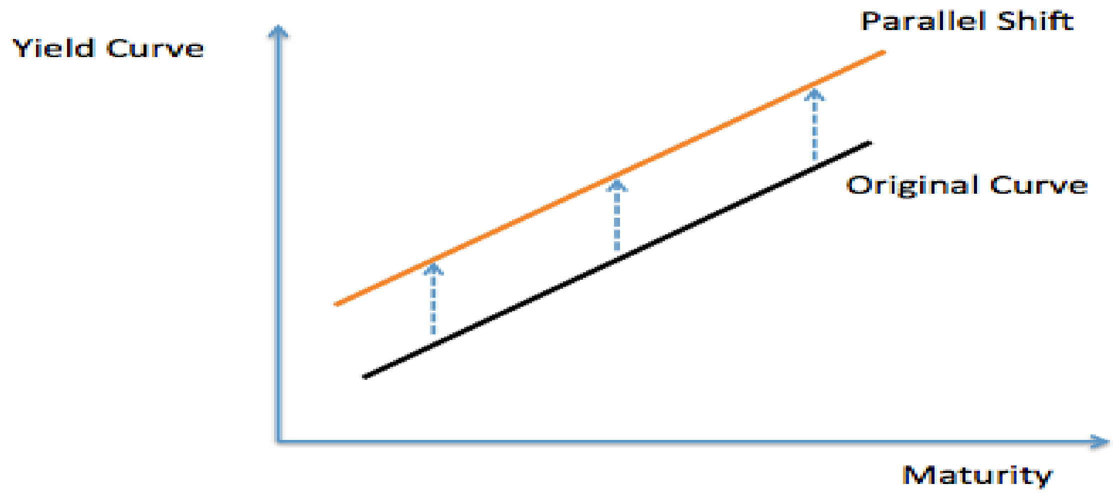

Figure 2. Parallel shift of the Yield curve.(source: www.financetrain.com)

So the major limits of using the Duration to define the volatility of the bond price are:

1. Inaccurate approximations for wide changes in the yield value;
2. The portfolio duration does not catch the non - parallel shifts of the yield curve.
3. The volatility of the yields is not taken into account.

Duration and convexity are not the only measures of the risk in fixed income portfolio. Increasing importance has the Value at Risk (VAR). This measure is very important especially for banks and financial institution because it was identified as benchmark for risk measures by Basel II committee. The first pillar of "The 1996 Amendment"[8] regards the "minimum capital request" and it states that a bank or regulated institution has to face three main kind of risks: credit risks, operational risks and market risks. The committee highlighted that the preferred approach to compute the last of these three risks is exactly the Value at Risk. The main characteristic of this value is that it summarizes the whole risk of the portfolio in one single number. The Value at Risk is defined by three parameters: the amount of loss "L", the likelihood to lose the amount L and the time horizon of the investment. There are two main method to compute the VaR: *historical simulation approach* and *model building approach*[5].

The historical simulation approach can be divided into the following steps:

- Identify the factors (or variables) that can influence the portfolio behavior;
- Generate a database including "n" daily variations of each factor. At this point "m" different scenarios are created considering only the "day 1" variations of each variables for Scenario 1, the "day 2" variations for Scenario2, and so on;
- Calculate the variation in portfolio's value for each scenario;
- Sort in ascending order all the variations. The probability of loss, defined by $1 - \alpha$, can be interpreted as the $1 - \alpha$ percentile of the loss distribution.

The *model building approach* or *variance-covariance approach* considers the daily variance of a security instead of the annual volatility (normally used in the option pricing processes). The relationship between these two measures is the following:

$$\sigma_a = \sigma_d\sqrt{252}$$

Where σ_a and σ_d are, respectively, the annual and the daily volatility of an asset and 252 are the working days per year. For instance, we can consider a portfolio composed by a single asset with a time horizon N=10 days and a confidence interval X= 99%. Assuming that Microsoft daily variance is equal to σ_d= 2% (so annual variance corresponds to σ_a= 32%) and value of the position equal to 10mil $, than a daily standard deviation of 2% on 10.000.000 is equal to 200.000 $. The table of the normal distribution associates 1% probability to the reduction of the variable's value that is equal to 2,33 times the standard deviation. So the daily VaR is equal to

[8] John C. Hull, 2012, "Option, Futures and other derivatives", 8 ed., global edition, Harlow : Pearson, pp.513-33.

2,33*200.0000= 465.270$ and the VaR of 10 days at 99% confidence is 465.270$*$\sqrt{10}$ = 1.471.312$[9].

The main difference between Duration and VaR approaches is that the latter is used to as a back test for the financial models and it is a measure to evaluate the performances of the portfolio manager ex post, while the former is portfolio strategy that the manager has to implemented in advance.

1.5 *Fixed income strategies*.

Fixed income portfolio strategies are divided in two big branches: *passive* and *active*.

Passive strategies.

With a passive strategy an investor takes a simple "buy and hold" position on the market. The strategy is achieved by purchasing a bond (or a quote of bond index fund) and holding it until maturity. Another option is the investment in mutual fund or exchange-traded fund who replicates bond market index. Passive approaches are indicate for investors who are interested in minimizing transaction costs, protect their capital and returns, but do not look for opportunities given by market conditions such as interest rates movements. The only "active" action they can take is to decide whether or not to reinvest the coupon cash flows in the portfolio or to use them to face other needs. The most common passive strategy is the *bond indexing strategy* which provides the creation of portfolio which aims to replicate the performances of a bond index. Those index are called benchmarks. Issues regarding benchmark selection are discussed in chapter 3. The most famous bond index used mainly by institutional investors are the "Salomon Brothers Broad Investment-Grade bond index" and "Merrill Lynch Domestic Market Index".

Then we can find more specific index like the "customized index" or "specialized index" where the former is create in order to meet the needs of the investors who want to reach some specific risk – return targets, while the latter are focused on specific sector or sub-sector of the market (corporate index, AAA sovereign, etc…). The bond index strategy provides some operational issues:

[9] John C. Hull, 2012, "Option, Futures and other derivatives", 8 ed., global edition, Harlow : Pearson, pp.515-16.

- If the benchmark is composed by a large number of bonds, it is difficult to replicate its return (high costs);
- Some bonds could not be available anymore;
- A fund that contains illiquid securities can reflect biased returns;
- The total return depends upon the coupon's reinvestment rate that can be overestimated by the fund.

The *Enhanced Indexing* strategy is very similar to the bond index strategy but it has the additional goal to outperform the target index with the aim of cover the management fees. In this case the benchmark's total return becomes the minimum target of the fund. As a result, it implies a little active management even though with low risk profile. For instance the manager can operate either with securities who are not included in the index or derivatives.

Active strategies.

An active manager has the target to outperform the selected benchmark by maximizing the total return of the portfolio. There are four factors whic can affect a portfolio return: interest rate movements, changes in the shape of the yield curve, yield spread variation between different sectors and changes in the risk premium for a specific bond. In order to choose the right way to outperform the selected index, an active manager has to create some expectation regarding the future behavior of interest rates. It is possible to distinguish four active strategies:

1) Interest rate expectation strategy.

This strategy operates on direction, dimension and timing of the interest rates movements. Based on the projections given by the models, the manager has two ways to operate depending on the direction of the movement:

- With rising interest rate expectations, the manager has to reduce portfolio duration;
- With falling interest rate expectations, the manager has to increase portfolio duration.

We saw before that a portfolio duration is the weighted average of single bonds duration. Thus, the portfolio composition must be modified in order to achieve the target duration. This can be done by:

- Buy or sell bonds;
- Buy or sell interest rate futures (more efficient because of the lower costs).

2) Yield curve strategy.

The yield curve strategy consists in optimizing expected variation of the yield curve by dividing the portfolio maturities in three sectors: short, medium and long term. Based on expectations the manager will change the weights on each sector. Three strategies are possible:

a. *Bullet strategy*: maturities are concentrated in one single point of the yield curve.

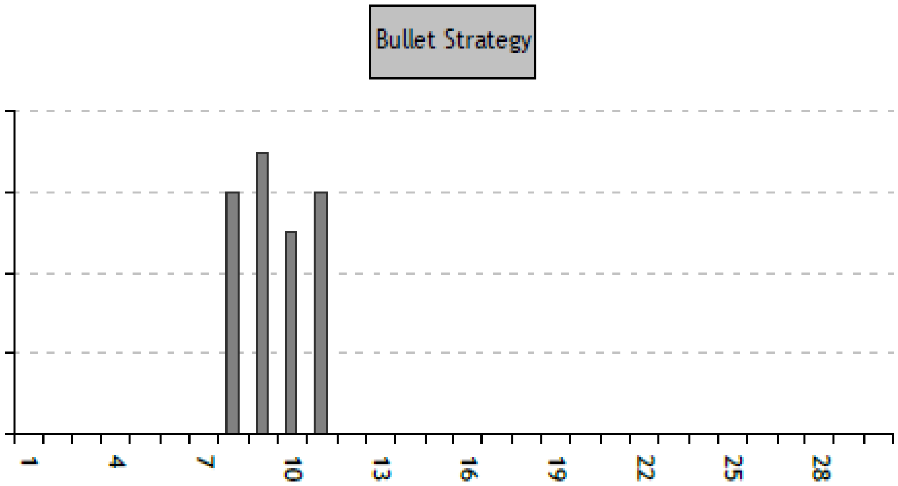

Figure 4. Bullet strategy. The portfolio composition is centered on year 9. source: (L. Cappellina, "Laboratorio di Finanza", pg. 508)

b. Barbell strategy: maturities are concentrated on two periods.

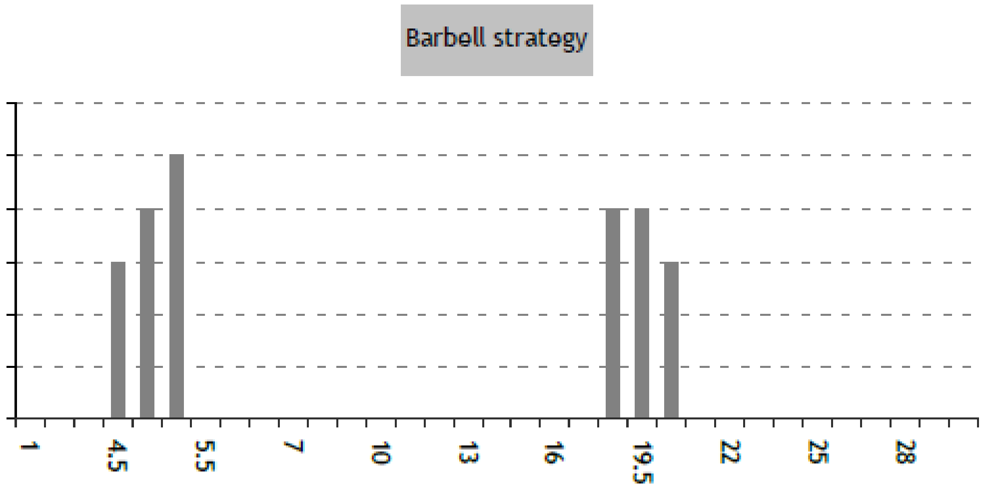

Figure 5. Barbell strategy. The portfolio composition is centered on year 5 andyear 19. (source: L. Cappellina, "Laboratorio di Finanza", pg. 508)

c. Ladder strategy: maturities are equally distributed over the time horizon.

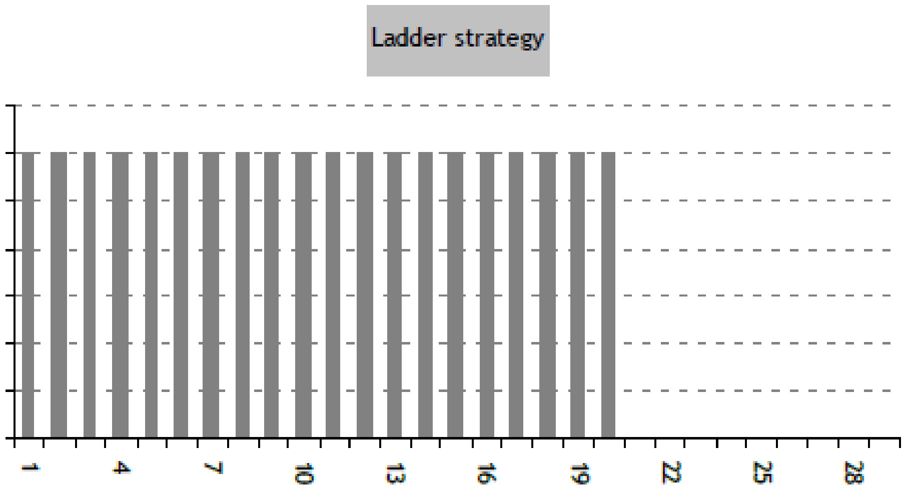

Figure 6. Ladder strategy. The portfolio composition is equally distributed over the all maturities (source: L. Cappellina, "Laboratorio di Finanza", pg. 510)

3) Yield spread strategy.

This strategy is connected to analysis of each bond based on:

- nature of the issuer (governments, corporate,…);
- credit quality (rating);
- coupon yield (high, medium, low);
- maturities (long, medium, short).

Spreads among bonds belonging to same or different sector can be a exploited to obtain higher return. There are three methods to calculate the yield spread:

- Absolute yield spread (yield A – yield B);
- Relative yield spread (yield A – yield B)/yield B;
- Yield ratio (yield A/yield B).

Credit spread between government and corporate bonds can be a predictor of recession/expansion periods when its value grows/falls.

4) Individual security selection strategy.

This strategy focuses on the identification of underestimated bonds that are likely to be upgraded by rating agencies in the future. Moreover, a fund manager can replace a bond with another having similar characteristics like coupon, maturity and credit rating, but generating higher returns.

We saw that the selection of the right strategy to invest in fixed income securities, in most cases, cannot be separated from the analysis of the yield curve. It is fundamental for the investor to create some expectation about the future behavior of the interest rates in order to choose the best timing of investment. Before starting the study of the various interest rate models in the next chapter, there are some other methods that are worth discussing. In particular investors can use four models to achieve this goal[10]:

[10] Fabozzi, Frank J. 1997 "Managing Fixed Incombe Portfolios", Frank J. Fabozzi Associates, pp. 149-153.

1. Fair value model.

   This models generate a fair value of interest rates that is compare with the current level; in this way it is possible to identify overestimated and underestimated bonds. This model made use of Fisher equation:

$$\text{Fair Value (FV)} = \text{real interest rate (RR)} + \text{expected inflation premium}$$

   The hypothesis underlying the Fair Value Model are:

   - Constant real interest return (RR),
   - Real interest spot rates converge towards their average over the time.

$$RR = Average\ (Y_{t,p} - CPI_{p+1})$$

   Where $Y_{t,p}$ is the nominal return for the maturity "t" at time "p" and CPI_{p+1} is the percentage expected variation of the Consumer Price Index on 12 months. The second element of Fisher equation, that is the expected inflation premium is not constant over time. It is determined by a 5 years moving average of annual inflation. Thus, the Fisher formula becomes:

$$\text{Fair Value (FV)} = \text{RR} + \text{5 years moving average of annual inflation}$$

   Once defined the FV, the model has to be implemented by estimating the magnitude of the difference between FV and observed interest rate and how long the bond can be under/over estimated and.

2. Bond Yield to Gold Model.

   The BYGM analyzes the link between inflation expectations and level of interest rates. It seeks to quantify the inflation expectations in a given period and use this information to estimate a appropriate level of interest rates. This theory suggests that gold will appreciate when rising inflation erodes the value of other investment instruments such as stocks and bonds (*flight to gold*). Consequently, gold can serve as a proxy for inflation expectations that are not incorporated in the interest rates.

3. Yield Curve Model.

   The Yield Curve Model states that the difference between thirty-year quarter yields of government bonds, defining the slope of the yield curve, can be used to predict the direction of interest rates in the long run. Empirical results show that the shape of the yield curve tends to converge to its mean. When the spread between the yield on thirty-year and quarterly that is higher than normal, long rates are expected to fall and bond prices rise to restore a normal differential. The model is a good predictor of changes in long-term trends, but not in the short-term market fluctuations.

4. The Macro-Pressure Model examines the major forces that are able to influence the macroeconomic scenario and how they affect domestic interest rates. It consists of three main components.

   - The first is related to the economy. This part of the model uses two indicators of inflation, a general index of future economic activity, and a measure of productivity (reflecting the strength of the economy).
   - The second component consists of two series that measure the performances of the recent financial markets.
   - The last component is made of two sets that measure the degree and direction of government intervention in the economy.

   The independent variables are adjusted to reflect their predictions over time and are then regressed against changes in year-over-year of the thirty - years yield (which is an estimation of the level of interest rates).

Chapter 2 - Short rate models.

2.1 Basics of short term models.

We saw the different shapes that the term structure can assume. But this description does not help us to forecast the future level of interest rates. For this purpose we need to produce an interest rate model. The short term interest rate is also called *instantaneous short rate* since it represents the rate at which a certain amount of money can be borrowed for an infinitely short period of time *t*. In a risk neutral world the actual value f_o of a future interest rate f_T, when $\bar{r}$ is the average value of r from t to T, is equal to[11]:

$$f_o = \hat{E}\left(e^{-\bar{r}(T-t)} f_T\right) \tag{2.1}$$

Where $\hat{E}$ represents the expected value of the elements inside the brackets. The same framework can be applied to the price of a bond. We denote with $P(t,T)$ the price of a bond who pays 1$ in T

$$P(t,T) = \hat{E}\left[e^{-\bar{r}(T-t)}\right] \tag{2.2}$$

Considering $R(t,T)$ as the continuously compounded interest rate at time t for the period $T-t$,

$$P(t,T) = e^{-R(t,T)(t-T)} \tag{2.3}$$

therefore we can find $R(t,T)$

$$R(t,T) = -\frac{1}{T-t} \ln P(t,T) \tag{2.4}$$

and rewrite

$$R(t,T) = -\frac{1}{T-t} \ln\left\{\hat{E}\left(e^{-\bar{r}(T-t)}\right)\right\} \tag{2.5}$$

Now, based on the value of the spot short rate and the risk neutral process for r, we can compute the entire term structure of the interest rate at general time t.

According to the expectation theory, the long term rates are determined by current and future short term rates. Thus, if we can model short term interest rate, we can in turn use the information to forecast long term interest rates. An interest rate model is a probabilistic description of how

[11] John C. Hull, 2012, "Options, futures, and other derivatives" 8 ed., page 758, global edition, Harlow Pearson, pp.757-758.

interest rates can change over time. In order to have a good interest rate model we need to incorporate some statistical properties of interest rate movements like *drift*, *volatility* and *mean reversion*. As we are looking only at short rates, we are dealing with the so called "one factor models".

The dynamic of the short rate can be describe by the following equation:

$$dr = bdt + \sigma dz$$

Where,

- dr is the change in the short rate;
- dt is the change in time;
- dz is a random process;
- σ is a volatility term;
- b is the drift term.

The change in the short rate is related to a random process dz. We need to make some assumptions:

- The random term z follows a normal distribution with mean zero and standard deviation equal to one $N\ (0,1)$;
- The change in the short rate is proportional to the value of the stochastic term dz, which depends on the value of the volatility term σ;
- The changes in the short rate in any two different intervals are independent.

The expected value of the change in the short rate is equal to the short term b. In the special case where b is zero and the variance is one, the expected value of the short rate is equal to its current rate and the standard deviation of the change in the short rate over an interval T is equal to $\sqrt{T}$. We can rewrite the equation above by making the drift term and the standard deviation depending on the level of the short rate:

$$dr = b(r)dt + \sigma(r)dz$$

The equation above is also known as It$\hat{o}$ process and it describes the one factor short rate model in a risk neutral world. This formulation is used in the so called "equilibrium models"; one factor hypothesis implies that all interest rates move in the same direction in each short period of time, but not with the same magnitudes. There are three model specifications regarding the computational form of drift term and standard deviation:

a) Rendleman and Batter model $b(r) = \mu r$; $\sigma(r) = \sigma r$
b) Vasicek model $b(r) = a(b - r)$; $\sigma(r) = \sigma$
c) Cox Ingersoll Ross model $b(r) = a(b - r)$; $\sigma(r) = \sigma\sqrt{r}$

2.2 Rendleman e Bartter model.

The Rendleman and Batter model states that the process followed by the interest rate r is a Brownian process:

$$dr = \mu(r)dt + \sigma(r)dz$$

The parameters μ and σ are constant, this means that the process for the short rate r is a Wiener process, that is a particular case of the Markov process. We can identify a Wiener process when the variable r satisfies the following properties:

- The variation of Δr within the interval Δt is equal to:

$$\Delta r = \varepsilon\sqrt{\Delta t}$$

- The values of Δr in any two intervals Δt are independent.

From the properties above, we can say that r follows a Markov process and Δr is normally distributed with mean 0, variance Δt and standard deviation of $\sqrt{\Delta t}$. For instance, the variation between $r(T)$ and $r(0)$ is equal to the sum of all variations of r in N small intervals Δt:

$$r(T) - r(0) = \sum_{i=1}^{N} \varepsilon_i \sqrt{\Delta t}$$

Where $N = \frac{T}{\Delta t}$ and ε_i for $i = 1, \dots, N$ are values independently selected from a standard normal distribution. Thus, the difference $r(T) - r(0)$ is normally distributed with:

- Mean $[r(T) - r(0)]$ = 0;
- Variance $[r(T) - r(0)]$ = $N\,\Delta t = T$;
- Standard deviation = $[r(T) - r(0)]$ = $\sqrt{T}$

When $\Delta t \to 0$ the evolution of r tends to be more jagged since the smaller the value of Δt, the larger the value of $\sqrt{\Delta t}$. This is a typical motion of stock prices and it is widely used in the Black and Scholes model. The main difference between interest rate and stock price is that the former tends to be pushed towards its mean in the long period while the latter does not.

This model is missing an important characteristic of interest rates that is called *mean reversion*: when the interest rate r is above/below its long term level, the mean reversion effect generates a negative/positive trend towards the reversion level. In the next part we are going to analyze models that take into account this important effect (Vasicek, Cox Ingersoll Ross and no arbitrage models).

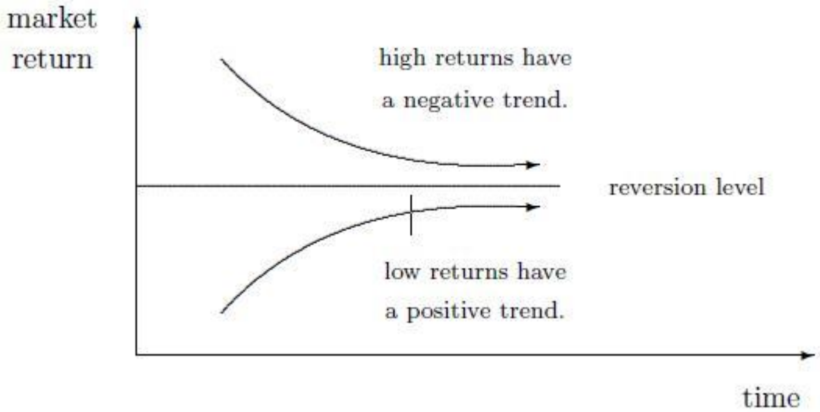

Figure 7. Mean reversion effect. (Source: http://www.tradingspotsilver.com)

2.3 The Vasicek Model.

This model was introduce by Oldřich Vašíček in 1977. It specifies that the short term interest rate follows the stochastic differential equation:

$$dr_t = a\,(b - r_t)dt + \sigma\, dz$$

Where

- a is the "speed" of reversion toward the mean;
- b is the long term interest rate level;
- $\sigma\, dz$ is the stochastic term with a normal distribution.

The Vasicek model is based on the following assumptions[12]:

[12] Vasicek, O.A., 1977, "An equilibrium characterization of the term structure of interest rates", Journal of Economics, page 178.

- The short term interest rate, r_t , follows a Markov process. This means that a future step s with $[(t,s), t \leq s\ ]$ depends exclusively on the current state t. Thus, the evolution of the spot rate over the interval depends only on r_t;
- The price of a zero coupon (discount bond) with maturity T is determined only by with $r(s)$ t <s <T;
- Efficiency of the market. Market efficiency means that there are not transaction costs, information is available to all investors, and that investors are acting rationally. Indeed, this means absence of arbitrage opportunities.

The second assumption implies that the price of a zero coupon evolves as

$$dP = \mu(t,s)Pdt + \sigma(t,s)Pdz$$

This expression is connected with the expression of r using Ito differentiation rule. Consider at time "t" to sell a quantity of bond W1 with maturity T1 and buy a quantity of bond W2 with maturity T2. The portfolio's value W = W2-W1 instantly evolve as

$$dW = (\mu(t,s_2)W_2 - \mu(t,s_1)W_1)dt + (\sigma(t,s_2)W_2 - \sigma(t,s_1)W_1)dz \tag{2.6}$$

By properly choosing the terms on the equation it is possible to eliminate the term dz, so that the portfolio becomes riskless. In particular we can use the following definitions:

$$W_1 = \frac{W\sigma(t,s_2)}{\sigma(t,s_1) - \sigma(t,s_2)}$$

$$W_2 = \frac{W\sigma(t,s_1)}{\sigma(t,s_1) - \sigma(t,s_2)}$$

At this point the equation (2.6) can be written as

$$dW = W[(\mu(t,s_2)\sigma(t,s_1) - \mu(t,s_1)\sigma(t,s_2)][\sigma(t,s_1) - \sigma(t,s_2)]^{-1}dt$$

Given the no arbitrage assumption we have

$$q(t,r) = \frac{\mu(t,s_1) - r(t)}{\sigma(t,s_1)} = \frac{\mu(t,s_2) - r(t)}{\sigma(t,s_2)}$$

“The quantity $q(t,r)$ can be called the market price of risk, as it specifies the increase in expected instantaneous rate of return on a bond per an additional unit of risk”[13]. This value is assumed to be constant over the time. Pricing a bond with those assumption leads to the equation:

$$\frac{\partial P}{\partial t}(ab + q_0 - ar)\frac{\partial P}{\partial r} + \frac{1}{2}\,\sigma^2\frac{\partial^2 P}{\partial r^2} - \,rP = 0$$

Vasicek demonstrated that, based on the equation (2.2), and considering a risk neutral environment, the price of a zero coupon bond at time t is equal to:

$$P(t,T) = \,A(t,T)e^{-B(t,T)r(t)} \tag{2.7}$$

With,

- $r(t)$: value of short term interest rate at time t

- $A(t,T) = \,e^{\frac{[B(t,T)-T+t](a^2b-\frac{\sigma^2}{2})}{a}-\frac{\sigma^2B(t,T)^2}{4a}}$ (2.8)

- $B(t,T) = \,\frac{1-e^{-a(T-t)}}{a}$

In the particular case of $a = 0$, we have:

- $A(t,T) = \,e^{\frac{\sigma^2(T-\,t)^3}{6}}$

- $B(t,T) = \,T - t$

The Vasicek model states that the instantaneous interest rate is pushed by the some forces towards its long-term mean γ with speed equal to a, against the randomness force produced by the stochastic term $\sigma\,dz$. This formulation represent a Ornstein-Uhlenbeck process that is a Markov process with normally distributed increments. It is different from the unstable Wiener process who tends to diverge to infinite values on the long run; instead the Ornstein-Uhlenbeck process is characterized by a stationary distribution.

[13] Vasicek, O.A., 1977, “An equilibrium characterization of the term structure of interest rates”, Journal of economics, page 180.

1.2 *The Cox, Ingersoll, Ross model*.

CIR model[14] is a reformulation of the Vasicek model. Indeed, the stochastic differential equation underlying this model is:

$$dr_t = a\,(b - r_t)dt + \sigma\sqrt{r}dz$$

The mean reversion effect is the same we saw in the Vasicek model, but the stochastic part is different. The standard deviation is multiplied by the square root of the short term interest rate; the rise of the short rate is proportional to the rise of the standard deviation increases. The assumptions underlying the model can be divided in two categories:

1. *Relating to the structure of the market*:
   - Competitiveness and the absence of frictions in the market;
   - Continuity of trade;
   - Infinite divisibility of assets;
   - Ability to lens/borrow any amount of money;
   - Short selling is permitted.
2. Related to the dynamics of the short rate r and to the preferences of the investors:
   - Agents have a logarithmic utility function;
   - The market price of risk is q supposed linear with respect to the square root of r;
   - The term structure depends only on the spot rate $r(t)$.

The authors showed that the price of a bond is the same as Vasicek's model[15]:

$$P(t,T) = A(t,T)e^{-B(t,T)r(t)}$$

But with different functions A and B:

- $A(t,T) = \left\{\frac{2\gamma e^{(a+\gamma)(T-t)/2}}{(\gamma+a)\left[e^{\gamma(T-t)}-1\right]+2\gamma}\right\}^{2ab/\sigma^2}$
- $B(t,T) = \frac{2\left[e^{\gamma(T-t)}-1\right]}{(\gamma+a)\left[e^{\gamma(T-t)}-1\right]+2\gamma}$

Where $\gamma = \sqrt{a^2 + 2\sigma^2}$

[14] Cox, J.C., J.E. Ingersoll and S.A. Ross (1985). "A Theory of the Term Structure of Interest Rates", *Econometrica*, 53: 385–407.
[15] J John C. Hull, 2012, "Option, futures and other derivatives" ", 8 ed., global edition, Harlow:Pearson, pp.757-758.

The mean reversion effect, given by the drift term $a\,(b - r_t)$, ensures a long run movement of the short rate towards the value b with a speed of a. The particular form of the standard deviation avoids the possibility to have negative interest rates. This interpretation is more consistent with the real world where we cannot find negative interest rates.

1.3 Properties of Vasicek and CIR models.

Functions A and B of Vasicek model are different from the measure in the CIR model, but they both move from the equation (2.6). We can obtain the equation for $R(t,T)$ by taking the first derivative with respect to $r(t)$,

$$\frac{\partial P}{\partial r(t)} = -B(t,T)P(t,T)$$

And based on the equation (2.4), we obtain:

$$R(t,T) = -\frac{1}{T-t}\ln[A(t,T)] + \frac{1}{T-t}\,B(t,T)r(t)$$

Using this formulation it is possible to determine the entire shape of the term structure as a function of $r(t)$. $R(t,T)$ depends linearly on the short rate; this relationship implies that the level of the term structure of interest rates at time t is determined by $r(t)$.

1.4 Calibration of the models.

The next step is to calibrate the models. This means we need to choose the values of a, b and σ that better characterize the term structure of a particular short rate. There are two different methods to face this problem[16]. The first method is based on historical dataset of short rate and it can be implemented by either:

- Construct a regression analysis of historical dataset for a given short term interest rate;
- Apply to the dataset a Maximum Likelihood analysis.

A second method consists in minimizing the sum of the square deviations between historical bond prices and theoretical bond prices provided by the estimations of the model. The main difference between these two methods is that the results are estimations on the real world in the first case, and estimations on the *risk neutral world*. The former is itself divided in: *Least Squares*

[16] John C. Hull, 2012, "Option, futures and other derivatives" ", 8 ed., global edition, Harlow:Pearson, pp.765-766.

regression method and *Maximum Likelihood method*. The calibration using least squares regression can be estimated using a linear regression of Δr on r

$$\Delta r = a(b - r)\Delta t + \sigma\varepsilon\sqrt{\Delta(t)}$$

Where the relationship between consecutive observations r_i and r_{t+1} is linear with an independent identically distributed random term ε.

While the maximum likelihood method solves the problem by computing the following likelihood function:

$$\sum_{i=1}^{m}\left\{-\ln\left(\sigma^2\Delta t - \frac{[r_i - r_{i-1} - a(b - r_{i-1})\Delta t]^2}{\sigma^2\Delta t}\right)\right\}$$

The risk neutral approach considers the same volatility found with the historical method, but the drift term is different. In particular it is reduced by the amount $\lambda\sigma$; therefore, the stochastic process followed by the short term interest rate r becomes:

$$dr = [a(b - r) - \lambda\sigma]dt + \sigma dz$$

That can be rewrite as

$$dr = a(b^* - r)dt + \sigma dz$$

Where

$$b^* = b - \frac{\lambda\sigma}{a}$$

The variable λ represents the market price of the interest rate risk. Ahmad and Wilmott have calculated that the market risk for the American market is $\lambda = -1.2$[17]; using the value b^* instead

[17] R. Ahmad and P. Wilmott, 2007, "The market price of interest rate risk: measuring and modeling fear and greed in the fixed income market.", Wilmott Magazine, 64-70.

of b in the equation (2.7) it is possible to determine the term structure of interest rates on different periods considered by Monte Carlo simulations.

1.5 No arbitrage models.

The weakness of equilibrium models (Rendleman and Bartter, Vasicek, CIR) could be identified on the no-perfect fit with the current term structure of interest rates. Indeed, these models give the term structure as an output, based only on the current level of the short term rate. Conversely, no-arbitrage models use the actual composition of the term structure as an input. Looking at the modeling part, the difference consists in the time depending of the drift term dt. No-arbitrage models consider the drift term as function of time and dependent on the short rate. Consider a period of time $[t_1, t_2]$: if the yield curve is upward sloping, the short rate will tend to increase; otherwise it will tend to decrease when the yield curve is downward sloping.

1.5.1 Ho – Lee model.

In their article "Term structure movements and pricing interest rate contingent claims" [18], the authors designed a model where the short rate dependence on the standard deviation σ of the short rate and the market price of the interest rate risk (in this model indicated with Θ). The model is:

$$dr = \Theta(\mathrm{t})\mathrm{dt} + \sigma \mathrm{d}$$

The standard deviation is supposed to be constant, while the variable $\Theta(\mathrm{t})$ is defined by:

$$\Theta(\mathrm{t}) = F_t(0,t) + \sigma^2 t$$

Where $F_t(0,t)$ is the partial derivative with respect to t and represents the instantaneous forward rate at time 0 with maturity t. The expected interest rate movement can be approximated to the slope (or direction) of the instantaneous forward rate curve (Figure 6). The price of a zero coupon bond can be calculated with the equation:

$$P(t,T) = A(t,T)e^{-r(t)(t,T)}$$

Where

$$ln[A(t,T)] = ln\left[\frac{P(0,T)}{P(0,t)}\right] + (T-t)F(0,t) - \frac{1}{2}\sigma^2 t(T-t)^2$$

[18] T.S.Y. Ho, S. B. Lee, 1986, "Term structure movements and pricing interest rate contingent claims", Journal of finance, 1011-29.

These equations compute the future price of a zero coupon in t as a function of the short rate at time t and the current prices of the zero coupon bonds.

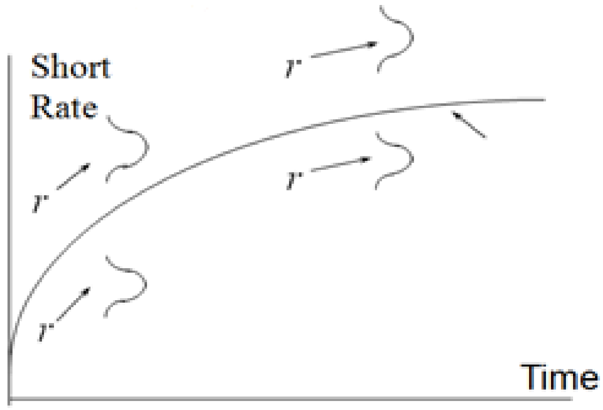

Figure 8. Ho-Lee model. **(Source: "Options, Futures, and Other Derivatives, 8th Edition, John C. Hull 2011")**

1.5.2 *Hull and White model.*

Hull White model is an extension of Vasicek model, it is constructed in order to include the perfect fit of the model with the current term structure[19]. The formulation is

$$dr = [\theta(t) - ar]dt + \sigma dz$$

or, equivalently

$$dr = a\left[\frac{\theta(t)}{a} - r\right]dt + \sigma dz$$

where the terms a and σ are constant. The Hull White model includes the mean reversion effect with a long term level of $\frac{\theta(t)}{a}$ and speed of reversion a. When the short rate goes too much above or under the yield curve, it tends to come back with a speed of a (Figure 7). The price of a zero coupon bond at time t is equal to:

$$P(t,T) = A(t,T)e^{-B(t,T)r(t)}$$

where

$$B(t,T) = \frac{1 - e^{-a(T-t)}}{a}$$

[19] J. Hull, A. White, 1990, "Pricing interest rate derivative securities", Review of Financial Studies, 3,4,573-92.

$$ln[A(t,T)] = ln\left[\frac{P(0,T)}{P(0,t)}\right] + B(t,T)F(0,t) - \frac{\sigma^2(e^{-aT} - e^{-at})^2(e^{2aT} - 1)}{4a^3}$$

These equations define the future price of a zero coupon in *t* as a function of the short rate at time *t* and the current prices of the zero coupon bonds.

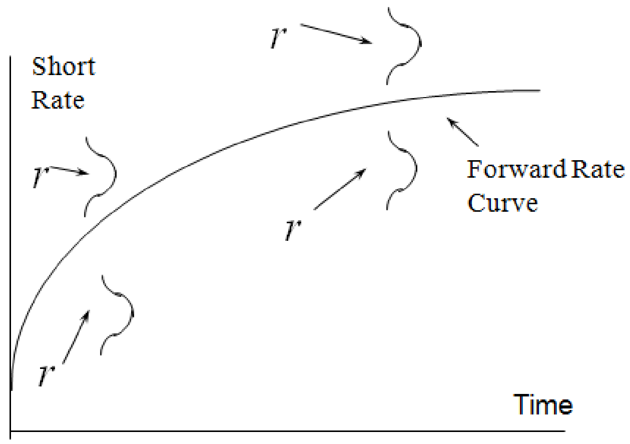

Figure 9. Diagrammatic representation of Hull-White model. (Source: "Options, Futures, and Other Derivatives, 8th Edition, John C. Hull 2011")

1.5.3 Black, Derman and Toy model.

The Black–Derman–Toy model (BDT) is a short rate model widely used in the pricing of bond options, swaptions[20]. It is the first model who aims to combine the mean reverting behavior of the short rate with the lognormal distribution of short term rate. The model is described by the equation:

$$dln(r) = [\theta(t) - a(t)\ln(r)]dt + \sigma(t)dz$$

where

$$a = -\frac{\sigma'(t)}{\sigma}$$

Unlike Ho-Lee and Hull-White models, the BDT model generates exclusively positive short term interest rates. The weak points of the model are:

- No analytical results;

[20] F. Black, E. Derman, W. Toy, 1990, "A One-factor model of interest rate and its application to treasury bond prices", Financial Analysts Journal, 33-39.

- The construction of the binomial tree for the short rate projections, is constrained to the relationship between $\sigma(t)$ and $a(t)$.

The most used interpretation of the model considers the term $\sigma(t)$ as a constant; this leads to the elimination of the parameter a and, in turn, the mean reversion effect is not considered. The model becomes:

$$dln(r) = \theta(t)dt + \sigma(t)dz$$

1.5.4 Black and Karasinski model.

This model is an extension of the BDT framework that aims to separate the determination of the reversion rate $\theta(t)$, from the volatility $\sigma(t)$[21]. The general version of this model is exactly the same as the BDT model

$$dln(r) = [\theta(t) - a(t)\ln(r)]dt + \sigma(t)dz$$

but on the practical application $a(t)$ and $\sigma(t)$ are taken as constants. The model then is written as:

$$dln(r) = [\theta(t) - a\ln(r)]dt + \sigma dz$$

The model is often used to evaluate exotic interest rate derivatives and swaptions. The function $\theta(t)$ is calibrated in order to fit the current term structure of interest rates. The model do not offer analytical results. The dynamic of short rate r can be represented by using a trinomial tree for interest rates.

1.6 Multifactor models.

One factor models described in this chapter are easy to implement but they face two major limits: they refer to only one risk factor (short interest rate) and they do not allow to choose the volatility structure. Among multifactor models there are the Longstaff and Schwartz model[22] and the Chen

[21] F. Black, P. Karasinski, 1991, " Bond and option pricing when short rates are log normal", Financial analysts Journal, 52-59.
[22] Francis A. Longstaff, Eduardo F. Schwartz, 1992. "Interest rate volatility and term structure: a two factor general equilibrium model", The Journal of Finance, 47, 1259-1282.

model[23] that are respectively a two factor model and a three factor model. The latter is sometimes called "stochastic mean and stochastic volatility model". The multi-factor framework is sometimes preferred over one-factor, because it can produce scenarios which are more consistent with current yield curve. Other famous models are the Heath, Jarrow and Merton (HJM) model[24] and the Libor Market Model (LMM)[25]. They allow for a free choice of both the term structure and the volatility.

[23] Lin Chen, 1996. "Stochastic Mean and Stochastic Volatility — A Three-Factor Model of the Term Structure of Interest Rates and Its Application to the Pricing of Interest Rate Derivatives", *Financial Markets, Institutions, and Instruments* **5**: 1–88.
[24] Heath, D., Jarrow, R. and Morton, A. ,1990. "Bond Pricing and the Term Structure of Interest Rates: A Discrete Time Approximation." *Journal of Financial and Quantitative Analysis,* 25:419-440.
[25] M. Musiela, M. Rutkowski, 2005. "Martingale methods in financial modelling", Springer, 2nd ed.

Chapter 3 - Optimization models for fixed income portfolios.

3.1 *Portfolio optimization. An overview.*

Portfolio optimization is a process that aims to obtain the best possible portfolio composition given a set of choices regarding the proportion of different assets. The criteria considered by an optimization model are usually the expected value of portfolio's rate of return and the variations over the time of these returns. Particularly, portfolio optimization problems are focused on the identification of portfolios which minimize the risk and match (or exceed) the return of the investment. A portfolio selection process specifies the weights to assign to each individual security. Markowitz model[26] is the landmark of portfolio optimization; it stated that the goal of the portfolio selection problem is to seek: minimum risk for a given level of return; maximum return for a given level of risk. Portfolios satisfying these criteria are efficient portfolios. They can be represented in a risks/returns graph where the horizontal axis indicates the risk (or volatility), while the vertical axis shows the expected return of the assets. They form a particular shape called the *efficient frontier* where each dot represents (Figure 6).

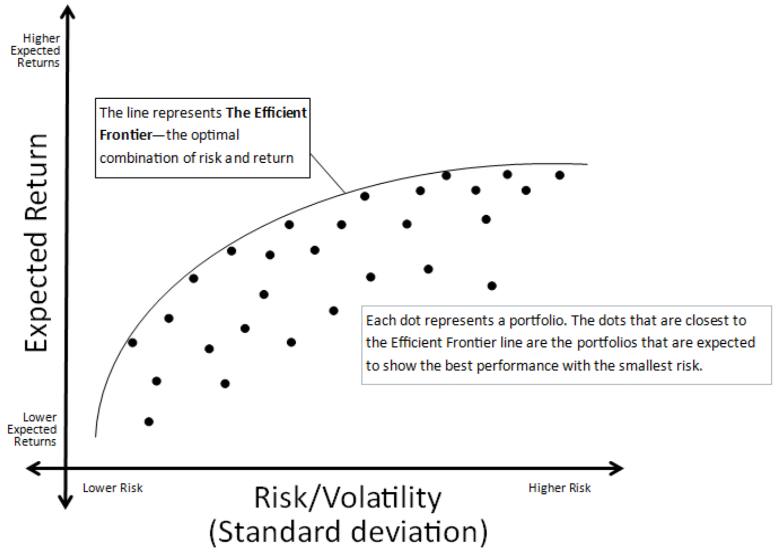

Figure 10. Graphical representation of the efficient frontier.

(Source: http://www.smart401k.com)

[26] Markowitz, H.M. (1959), "Portfolio Selection: Efficient Diversification of Investments.", John Wiley & Sons, Inc., New York, Chapman & Hall.

The risk can be decomposed in two broad categories: *Systematic risk* or *market risk*, is the risk which is common to all securities in the market. This type of risk cannot be diversified while we are constructing a financial portfolio. *Idiosyncratic risk or non systematic risk*, it is the risk referred to a particular asset and it can be eliminated through portfolio diversification.

The risk is measured as the likelihood that the value of a specific investment will raise or fall as the time passes and by which magnitude. We assume that investors prefer to minimize risk, so given two portfolios with equal returns, investors will choose the one with lower risk. The process of reducing the idiosyncratic risk is called diversification. It provides a split of the investment to a variety of assets. Since market risk cannot be controlled, diversification tools are designed exclusively to decrease idiosyncratic risk. The correlation between a variety of asset classes and sectors can lower the risk of expected losses when a particular type of security falls in value. In a static framework, with no rebalancing, assumed the risk aversion of the investors, the model introduced by Markowitz can be represented the following formulation:

$$E(R_p) = \sum_{i=1}^{N} w_i * E(R_i)$$

$$\sigma_P^2 = \sum_{i=1}^{N} w_i^2 \sigma_i^2 + 2\sum_{i=1}^{N} \sum_{j=i+1}^{N} w_i w_j \sigma_i \sigma_i \rho_{ij}$$

Where

- $E(R_p)$ = portfolio expected return;
- σ_P^2 = variance of portfolio returns;
- N = number of assets in portfolio;
- w_i = weight of asset i with i=1,…,N;
- σ_i = standard deviation of asset i;
- ρ_{ij}= correlation between returns of assets (i, j).

Given these assumption a possible formulation of the problem is:

$$\min(\sigma_P)$$

Subjet to: $$\sum_{i=1}^{N} w_i * E(R_i) = \mu = E(R_p)$$

$$\sum_{i=1}^{N} w_i = 1$$

Markowitz's approach has been widely applied to equity markets but much less can be founded on fixed income securities. This lack of applications can be explained by two reasons[27]:

1) During 1950s, when Markowitz's approach became popular, interest rates were not particularly volatile and, in turn, there was not a risk perception regarding bond markets. The situation changed over the years and changes in interest rates became one of the main risk to take into consideration when investing on bond markets. The result is that today diversification is applied also to manage fixed income portfolios;
2) A second reason is related to the technical difficulties in implementing Markowitz's approach. They are mainly referred to[27]:
    - The large number of parameters needed when the number of assets increases;
    - The variation of moments over time, which precludes simple historical estimation based on the assumption of stationarity.

Various authors attempted to adapt mean-variance analysis to bond portfolios. In the 1960s the topic was treated by Cheng (1962)[28], then during the 1970s researches were conducted by Roll (1971)[29], Yawitz, Hempel and Marshall (1976)[30], Yawitz and Marshall (1977)[31], and Kaufman (1978)[32].

3.2 General optimization models for fixed income portfolios.

Financial and nonfinancial institutions are often facing the problem to plan future liabilities cash flows. Sometimes they are known with certainty as for insurance/pension funds payments or, considering nonfinancial institutions, the acquisitions plans or product development programs. The manager target is to combine over the time these future outflows, with the incomes generated

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[27] Korn, O., Koziol, C. (2006). Bond Portfolio Optimization: A Risk-Return Approach, Journal of Fixed Income, 15(4), pp. 48-60.
[28] Cheng, P. L. (1962). Optimum Bond Portfolio Selection. Management Science, 8(4), pp. 490-499.
[29]R. Roll, "Investment diversification and bond maturity.", Journal of Finance, 26, pp.51-66.
[30] Yawitz, J. B., Hempel, G. H., Marshall, W. J., 1976, "A risk-return approach to the selection of optimal government bond portfolios.", Journal of Financial Researches, 5, pp. 171-180.
[31] Yawitz, J. B., Marshall, W. J., Journal of Portfolio, 1977, "Risk and return in the government bond market.",pp.36-45.
[32] Kaufman, G. G., 1978, "Measuring risk and return for bonds: a new approach.", Journal of Bank Research, 9, pp. 82-90

by the assets. There are two main strategies that aim to control the mismatch arising between assets and liabilities: Portfolio dedication and Portfolio immunization.

Portfolio dedication.

The goal is for this strategy is to match the timing of cash inflows to a predictable stream of cash outflows. It is sometimes called "cashflow matching". It aims to determine the least expensive combination of bonds looking at the right quantities and maturities that are able to match the cash flows. Suppose that at time t = 0, the manager has to match a stream of liability obligations of amount *l*, to be paid at time *t*. Assume that liability cash flows are fixed and certain and there is a finite amount of resources. The formulation of this problem is

$$\min \sum_k n_k p_k$$

Subject to

$$\sum_k n_k c_{k,t} \geq l_t \ for\ all\ t$$

and

$$n_k \geq 0\ for\ all\ k$$

Where:

- n_k is the number of units of security k purchased;
- p_k denote the current price for one unit of the k-th security;
- c is the cash flow at time t.

Portfolio immunization.

The weak point of portfolio dedication models is that they assume fixed reinvestment rate of the cash flows. The literature then stated to focus on risk management and passive immunization strategies [Fabozzi (2004)][33]. When interest rates change, the value of all cash flows will change. "Immunization strategy matches the interest rate risk of portfolio securities against the projected stream of liabilities to achieve a zero net market exposure"[34]. The most common way to think about the portfolio immunization it to consider a zero coupon bond with duration of D years

[33] Fabozzi, F. J., 2004, "Bond markets, analysis and strategies." Pearson Prentice Hall, Upper Saddle River.

[34] Stavros A. Zenios, 2007,"Practical financial optimization. Decision making for financial engineers." Hand book, Blackwell.

against a future financial obligation of F. An investor can buy a zero coupon bond with present value P which will pay the face value F after D years. Usually the specific zero coupon bond is not available, so the manager has to invest in a bond portfolio. Assuming that there are two bonds available Bond 1 (P_1, D_1) and Bond 2 (P_2, D_2) and one obligation F (with a present value of P). The required conditions for bond portfolio immunization are:

1. The weighted duration of the bonds should be greater than or equal to the duration of the obligation,

$$D \geq \frac{(D_1 + D_2)}{2}$$

2. The present value of the obligation is

$$P = \frac{F}{(1 + \frac{\lambda}{m})^{(D*m)}}$$

Where the term *m* indicates the number of coupons that have to be paid in one year period and λ is the yield apply by the market conditions. The manager then has to buy 'x' number of Bond 1 and 'y' number of bond 2 such that:

$$x * P_1 + y * P_2 = P \quad (3.1)$$

Considering the weighted duration we can write

$$x * P_1 * D_1 + y * P_2 * D_2 = P * D \quad (3.2)$$

The above equations state that future obligation F with a present value of P and duration D, can be matched by investing in 'x' Bond 1 and 'y' Bond 2, subject to the constraints (3.1) and (3.2). The same formulation can be extended to multiple bond selection.

3.3 Tracking error models.

The general optimization models are useful for simple problem classes. In order to consider a larger amount of risk factors, it is necessary to use measure of risk that can be applied across different asset classes. The concept underlying this interpretation is based on the fact that the goal of many investors is to replicate a certain benchmark return. In this case the risk perception is not related to maturities mismatching or interest rates per se, but it is referred to the volatility of the difference between portfolio returns and benchmark returns. Optimization models are used to minimize this measure that is also known as *tracking error volatility*. It can be interpreted in many different ways:

The mean squared approach is constructed to minimize the sum of the squared deviations of portfolio returns from a benchmark. The problem can be written as:

$$\min_{\beta} \varepsilon'\varepsilon = \min_{\beta}(Y - X\beta)'(Y - X\beta)$$

where

- X = continuously compounded returns on n assets;
- Y = vector of continuously compounded benchmark returns;
- β = portfolio weights;
- $\varepsilon = (Y - X\beta)$ deviation between benchmark and portfolio returns.

And the vector of assets weights is given by

$$\beta = (X'X)^{-1}X'Y$$

The popularity of the mean square model is due to:

1. computational simplicity;
2. the estimator β is an estimator BLUE (Best Linear Unbiased Estimator).

It is also possible to include additional restrictions addition to a set of linear restrictions can be included such as short selling who imposes positive portfolio weights and full investment of the capital who equals to one the sum of the weights. The model formulation is

$$A\beta \geq b$$

Where vector b represent the constraints applied to the model. The reason why quadratic tracking error measures is common in practice, is that they have desirable statistical properties. The weakness of this method is that quadratic objective functions are difficult to interpret.

Absolute deviation approach defines the tracking error as the absolute difference between the portfolio return and the benchmark portfolio. It was introduced by Clarke and others (1994)[35] and it is based on the fact that investors deal with investment objectives where absolute deviations between portfolio and benchmark returns are more relevant or have a more intuitive interpretation. In turn, the more reasonable way to model this framework is a linear programming approximation suggested by Sharpe (1971)[36] and Konno and Yamazaki (1991)[37] who developed a portfolio

[35] Clarke, R.C., Krase, S., Statman, M., 1994, "Tracking errors, regret, and tactical asset allocation.", The Journal of Portfolio Management, pp.16-24.

[36] William F. Sharpe, 1971, "A linear programming approximation for the general portfolio analysis problem.", Journal of Financial and Quantitative analysis, 6, pp. 1263-1275.

optimization model based on mean absolute deviations instead of the volatility of the portfolio returns. The main advantages of linear tracking error models are:

- Performance feed given to portfolio managers are based on the return difference between the portfolio and the benchmark (see Kritzman, 1987)[38].
- The attempt of managers to reduce or eliminate extreme deviation from the benchmark imply that managers prefer to think in terms linear deviations instead of quadratic.

There are several definition of tracking error. In their work Rudolf, Wolter, Zimmermann (1998)[39] analyzed four alternative tracking error definitions. The link between those interpretation is that tracking errors are based on linear objective functions with absolute deviations between portfolio and benchmark returns.

Mean absolute deviations (MAD) model.

The mean absolute deviation is the average distance of each element of the dataset from the mean of the same dataset. It represent on average how far each element or each information is away from the average of the dataset. So the MAD model minimizes these absolute deviations of portfolio returns from the benchmark by computing the optimal weights to apply to each security. The formulation is

$$min_{\beta}\ 1'(|X\beta - Y|)$$

Where $1' = (1,1 \ldots.,1)$. The advantage of this formulation is that the tracking error is measured as percentage, whereas if we consider the mean square objective function, the tracking error is measured as squared percentages. The MAD indicates how the dataset is spread. Large MAD value indicates large variations from the mean and in turn a wider risk. On the opposite a small MAD value indicates small variations from the means lower risk for the investor. We remind that the risk perception is the distance of portfolio return from benchmark return that is considered the "zero-risk value" but this does not imply that portfolio returns who perfectly match benchmark returns are riskless.

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[37] H. Konno, H. Yamazaky, 1991, " Mean-absolute deviation portfolio optimization model and its applications to Tokyo Stock Market.", Management science, 37, pp.519-531.

[38] Kritzman, M.P., 1987,"Incentive fees: Some problems and some solutions.", Financial analysts Journal, 43, pp.21-26.

[39] Markus Rudolf, Hans Wolter, Heinz Zimmermann, 1997, "A linear model for tracking error minimization.", Journal of banking and finance, 23, pp. 85-103.

MinMax models.

In this model, portfolio weights are determined in order to minimize the maximum deviation between portfolio and benchmark returns. The objective function of the MinMax optimization problem is

$$min_{\beta}\ (max_t |X_t\beta - Y_t|)$$

Where

- X_t= row t of matrix X
- Y_t = t-th element of vector Y.

An issue raised by Amemiya (1985) [40] is that outliers, that are the values much farther away from the mean with respect to the others, are squared in quadratic models so large deviations tends to have higher weight with respect to the MAD model. This characteristic can be translated in a lower sensitivity of MAD models against outliers than the mean square models.

Mean absolute downside deviation model (MADD).

MAD and MinMax models consider both positive and negative deviations as a measure of risk. This is typical for passive management strategies where the goal of investors is to replicate a benchmark return regardless they are positive or negative. On the opposite, an active manager considers a different perception of risk that appears only when portfolio returns are lower than benchmark returns, since positive differences represent a better performance. Harlow (1991)[41], described this problem by restricting the optimization framework only to the negative deviations between portfolio and benchmark returns, focusing the attention on the so called ``downside risk" of the investment. The formulation of this model is

$$min_{\beta}\ 1'(|X\beta - Y|)$$

Where

$$X_t\beta \leq Y_t$$

Downside MinMax model.

The same principle of MADD model can be applied to MinMax models. DMinMax model minimizes the maximum negative deviation between portfolio and benchmark returns:

[40] Amemiya, T., 1985, "Advanced Econometrics.", Harvard University Press, Cambridge, MA.
[41] Harlow, W.V., 1991, " Asset allocation in a downside risk framework.", Financial Analysts Journal , 5, pp.10-11.

$$min_{\beta}\ (max_t|X_t\beta - Y_t|)$$

Where

$$X_t\beta \leq Y_t$$

3.4 Linear programming.

Each tracking error definition discussed in the previous section can be described by a linear program:

(a) Mean absolute deviation (MAD). Given a positive deviation z^+ and the absolute value of a negative deviation z^- we can say that

$$X_t\beta - Y_t > 0 \leftrightarrow X_t\beta - z^+ = Y_t$$

$$X_t\beta - Y_t < 0 \leftrightarrow X_t\beta + z^- = Y_t$$

And the objective function becomes

$$\min \sum_{t=1}^{T} (z^+ + z^-)$$

With this formulation positive deviations z^+ implies that z^- is equal to zero, whereas negative deviations z^- implies that z^+ is equal to zero. These restrictions can be summarized in one single equation:

$$X_t\beta + z^+ - z^- = Y_t$$

(b) MinMax program. Given $z \geq 0$ the upper limit of the absolute deviation

$$z \geq |X_t\beta - Y_t| \qquad t = 1, \dots, T$$

Then the two cases are

$$z \geq X_t\beta - Y_t \geq 0 \leftrightarrow X_t\beta - z \leq Y_t$$

$$-z \leq X_t\beta - Y_t \leq 0 \leftrightarrow X_t\beta + z \geq Y_t$$

Where in the first case the portfolio return is higher than the benchmark return, and vice versa. The MinMax optimization is:

$$\min z$$

Subject to

$$X_t\beta - z \leq Y_t$$

$$X_t\beta + z \geq Y_t$$

(c) Downside MinMax program. In this case the MinMax problem is restricted to observations in which the portfolio returns are lower than benchmark returns,

$$X_t\beta - z \leq Y_t$$

Therefore we have:

$$\min z$$

Subject to

$$X_t\beta + z \geq Y_t$$

(d) Mean Absolute Downside Deviation (MADD). As we saw above, manager are concerned about negative deviations from the benchmark z^-, so the model is written in order to drop positive deviation z^+:

$$\min \sum_{t=1}^{T} z^-$$

Subject to

$$X_t\beta + z^- \geq Y_t$$

3.5 Benchmark selection.

Benchmarking is a process used to measure and compare financial performance against competitors. With the huge variety of benchmarks, decision making process is not easy. However this call is critically vital for many reasons:

- Portfolio risk and returns are going to be heavily influenced by the benchmark. Once portfolio managers have constructed a portfolio, they generally take the securities within the benchmark to start the implementation of active strategies with the goal to achieve better risk/returns targets.
- The benchmark indicates both the types of securities that should be enclosed and the types of securities that should not be enclosed to the portfolio. As an example, sovereign bond index could be a good estimator for fixed income portfolios. This choice is typical for investors which take low risk positions. It is the same in the stock market when an investor selects a low cap index, like the Russell 2000. This behavior suggests that investor wants to avoid the exposures towards large companies.
- Third, some benchmarks suited better than others to specific investment goals. For instance, an investor who pursues capital preservation may focus his attention on credit

ratings. Whereas an investor who operates in order to cover future liabilities (see 3.2) may focus his attention choosing a benchmark with the lowest rate sensitivity (or duration).

We can distinguish between three different kinds of benchmarks[42]:

1. Market index or market portfolio. The initial portfolio is specified by a weighted average of the prices of selected bonds index who represent a specific asset class. They are simulated portfolios (so transaction costs are not considered), but they are used as a proxy for measuring the value of a section of the market. Bond indices can be categorized based on some characteristics. The most famous are:

- Global bond :Barclays Capital Aggregate Bond Index;
- U.S. bond: Barclays Capital Aggregate Bond Index, Salomon BIG, Merrill Lynch Domestic Master, CPMKTB - The Capital Markets Bond Index;
- Government bond: Salomon Smith Barney World Government Bond Index, J.P. Morgan Government Bond Index, Access Bank Nigerian Government Bond Index, FTSE UK Gilts Index Series, MAX Hungarian Government Bond Index Series;
- Emerging market bond: J.P. Morgan Emerging Markets Bond Index, JPMorgan GBI-EM Index;
- High-yield bonds: CSFB High Yield II Index (CSHY), Merrill Lynch High Yield Master II, Bear Stearns High Yield Index (BSIX);
- Leveraged loans: S&P Leveraged Loans Index;
- Asset backed security: ABX index.

2. The second type of benchmarks are real portfolios whose returns, after costs and transaction expenses, are gathered for comparable portfolios. Samples of such portfolio lists are:

- Morningstar's short-term, high-quality bond index;
- Intermediate - term, low-quality bond index;
- Intermediate-term, intermediate-quality.

3. As opposed to utilizing asset portfolios as a benchmark, the liabilities may be utilized as the benchmark. This is the third sort of benchmark. It is one specification of portfolio dedication

[42] Frank J. Fabozzi, L. Martellini, P. Priaulet , 2006, "Advanced bond portfolio management.", hand book, John Wiley & Sons, Inc., Hoboken, New Jersey.

models. For instance, pension fund managers, since the poor profits of the sector after 2000, are starting to utilize the computed profit for their pension liabilities as a benchmark.

Another way to construct a benchmark, when it cannot be found on the real market, is to equally split the weights of each asset included in the index. The benchmark portfolio in this case is called equally weighted portfolio. Weighted portfolios have started to be examined by more researchers. Benartzi and Thaler (2001)[43] examined total distributions to different funds in the United States, and they appeared to demonstrate that investors utilized just as weighted holding strategies. Demiguel, Garlappi, and Uppal[44] looked at the return of a equally weighted portfolio with the returns of portfolios created by other generally utilized allocation models. They find that the weighted average methodology is "not extremely inefficient" and "performs well out of the sample".

Selecting the right benchmark by could be as essential as the individual investment choices. In the event that an improper benchmark is chosen with respect to the objectives of the fund, the supervisor may perform well against the index, however miss the mark regarding the wanted level of return of the fund. In the current environment there are lot of index suppliers, each with an alternate set of qualifying criteria characterizing the business sector. There are some generally standards of what constitutes a good index[45]:

Principle 1: Relevance to the investor.

Any index chosen as a benchmark should be a relevant investment for the investor. Relevancy means to avoid a “natural concentration” between the business risk of the index and the endowed portfolio. For this purpose several investors use custom indices. An example is the selecting process of an acceptable benchmark for a pension fund. A pension fund manager may need to use a portfolio of liabilities as a benchmark.

Principle 2: Representative of the Market.

A good benchmark should give a correct image of the market that is it pretends to represent. For instance, if the index size threshold is too high to catch a significant number of securities within

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[43] S. Benartzi and R.H. Thaler, 2001, “Naive Diversification Strategies in Defined Contribution Savings Plans", American Economic Review, 91.

[44] V. Demiguel, L. Garlappi, and R. Uppal, 2005, “How inefficient are simple asset allocation strategies?”, working paper.

[45] Frank J. Fabozzi, L. Martellini, P. Priaulet , 2006, “Advanced bond portfolio management.”, hand book, John Wiley & Sons, Inc., Hoboken, New Jersey.

the selected market, the performance of the index will be totally different from the performance of the market.

Principle 3: Trasparent Rules and Consistent Constituents.

We have several definitions of a bond index. One of those is that a bond index is a rules-based assortment of bonds. It is, therefore, imperative that the principles process the index are clear and objectively applied.

Principle 4: Investible and Replicable.

An investor ought to be ready to replicate the index and its performance with a tiny low range of instruments, with comparatively transaction costs and without moving the market for an excessive amount. For this reason the index constituents ought to be a collection of bonds that are liquid and traded actively within the secondary market. Indices with higher threshold levels usually contain fewer illiquid instruments and they are so simple to track for obvious reasons. This is due to the presence of a liquidity premium. In fact, bonds which are more liquid tend to be traded at tighter levels with respect to bonds that are less liquid. The lower yields applied to liquid bonds tend to generate lower returns, and in turn for managers will be easier to outperform the index.

Principle 5: High Quality Data.

An index prices and statistics are good only if the data used to compute them are calculated in the right way. Using biased prices, even for well constituted index, could lead to a distorted representation of the market. This can be an issue for over-the counter (OTC) markets where most of the bond indices use internal pricing method that are very likely to be distorted. It is then fundamental to avoid these distortions by ensuring that index pricing comes from a reliable source.

Principle 6: Independence.

Independence of the indices create the conditions for investor to select reliable data from multiple sources. This leads to the creation of “after-index products” such as derivatives. The presence of many investors active in the market ensures a competition that is good the development of the trading system.

Chapter 4 – Active strategy using short rate models. An empirical application.

4.1 Data description.

The goal of this work is to apply an active management strategy to a financial portfolio composed by fixed income securities. Indeed we are going to consider a portfolio composed by three Italian sovereign bonds with maturities of 10, 3 and 1 year. The short term interest rate is calibrated using Euribor interest rate dataset collected from 2000 to 2011 (Figure 7). Euribor is the acronym of "Euro InterBank Offered Rate" and it represents the average interest rate at which the so called "panel banks", that is a selection of different banks operating in European countries, lend and borrow money to each another denominated in Euros. The Euribor interest rate is a sum of 8 different rates with different maturities whom range between one week to 12 months. Likewise the vast majority of interest rate model applications, in this work we consider monthly observations of 3 months Euribor interest rate and we use the projections given by the models to apply an active management strategy for the year 2012. As we can see in Figure 7, during the last 11 years the blue line representing the Euribor interest rate followed a non linear path, characterized by two big humps.

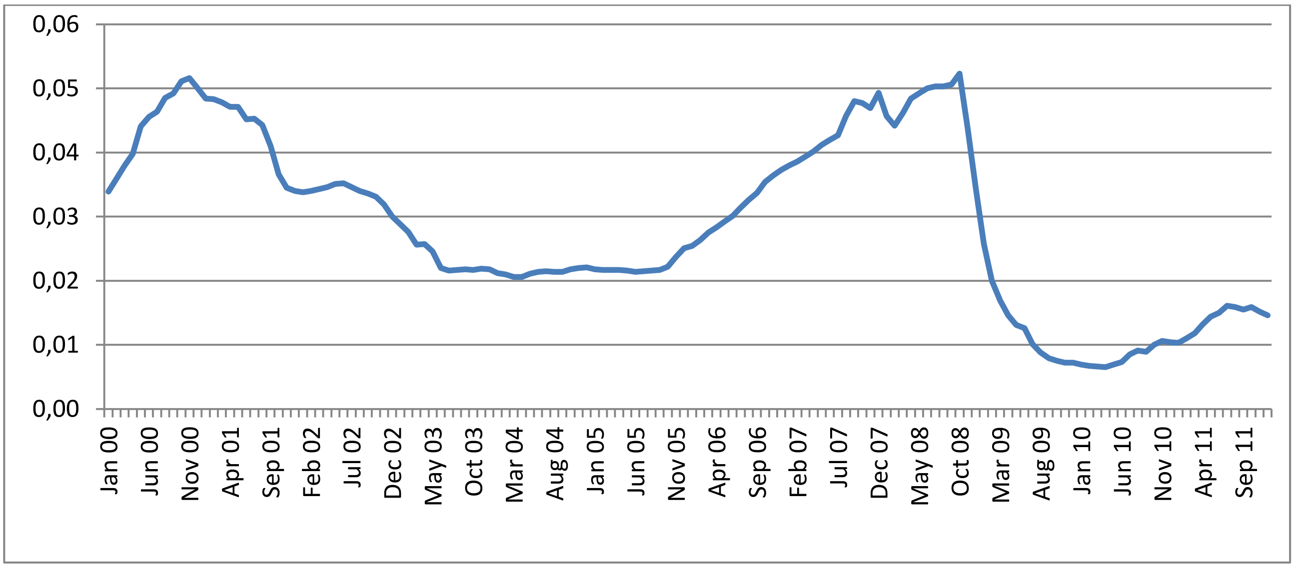

Figure 11. Monthly observations of 3 months Euribor rate from 2000 to 2011.

They can be linked to financial crisis which hit the European markets. The lack of trust among banks leads to widen of interest rates in order to compensate for the additional (default) risk perceived by the market. Using another graph we can see how volatile this index was during 2000s. We simply took the first differences between t and $t+1$, in order to show the monthly variability of the Euribor interest rates (Figure 8).

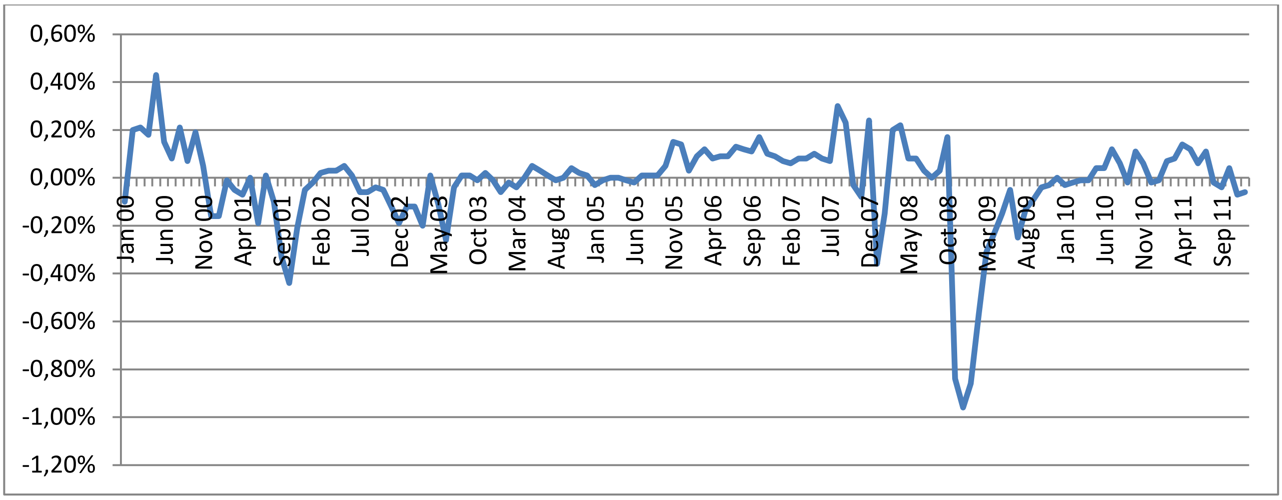

Figure 12. Monthly percentage variations of 3 months Euribor rate from 2000 to 2011.

Furthermore, we considered for the same time horizon the yields of Sovereign Italian zero rate bonds with 10, 3 and 1 year maturity. Figure 9 shows how the bond yields follow the 3 months Euribor interest rate. It turns out that in some specific periods, the one year bond yield is higher than the Euribor rate.

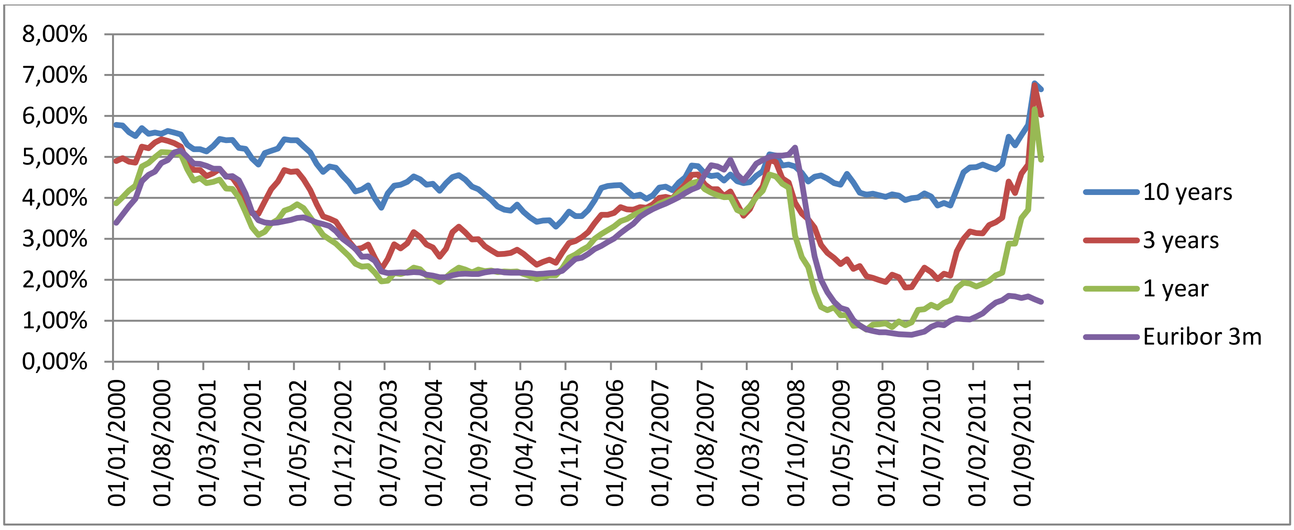

Figure 13. Sovereign Italian bonds yields at 10, 3 and 1 year, compared the 3 months Euribor interest rate for the period 2000-2011.

This means that banks were lending and borrowing each other by applying higher interest rate with respect to the Italian Government bond emissions. This spread is consistent with the action taken by the European Central Bank (ECB) who cut interest by a total of 325 basis points between October 2008 and May 2009 with an exceptional monetary policy measures. In turn, Eurozone Governments were able to issue bonds with higher interest rates, exploiting the very low yield requested from the ECB. The private sector is less flexible and it tends to react slower to monetary policy actions. Another interesting observation regards the inverse relationship between bond prices and interest rates (Chapter 1). Looking at the Euribor trend we can suggest that during the time intervals from January 2000 to January 2006 and from May 2008 to November 2009, the prices of all bonds should raise due to a fall in interest rate levels. As we can see in Figure 10 and in Figure 11, the black lines representing a linear trend for each of the time series are upward sloping, confirming our initial suggestions.

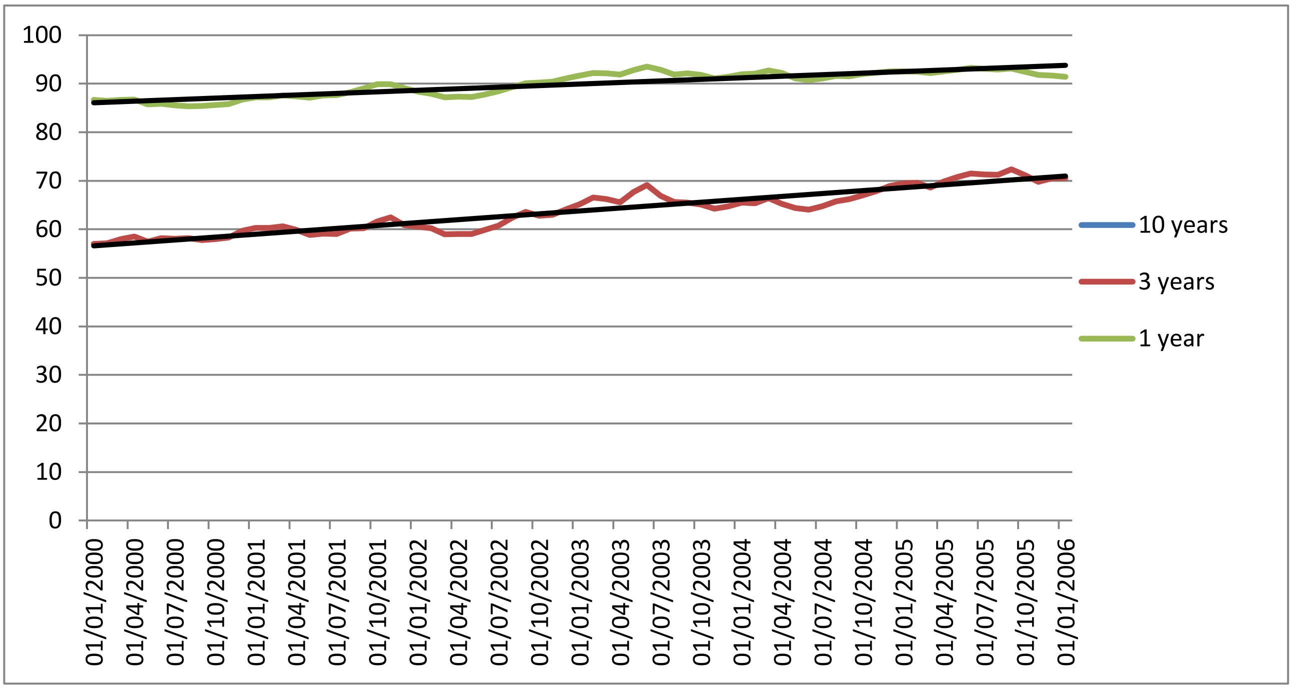

Figure 14. Positive trend of Sovereign Italian bonds prices for the period January 2000-January 2006.

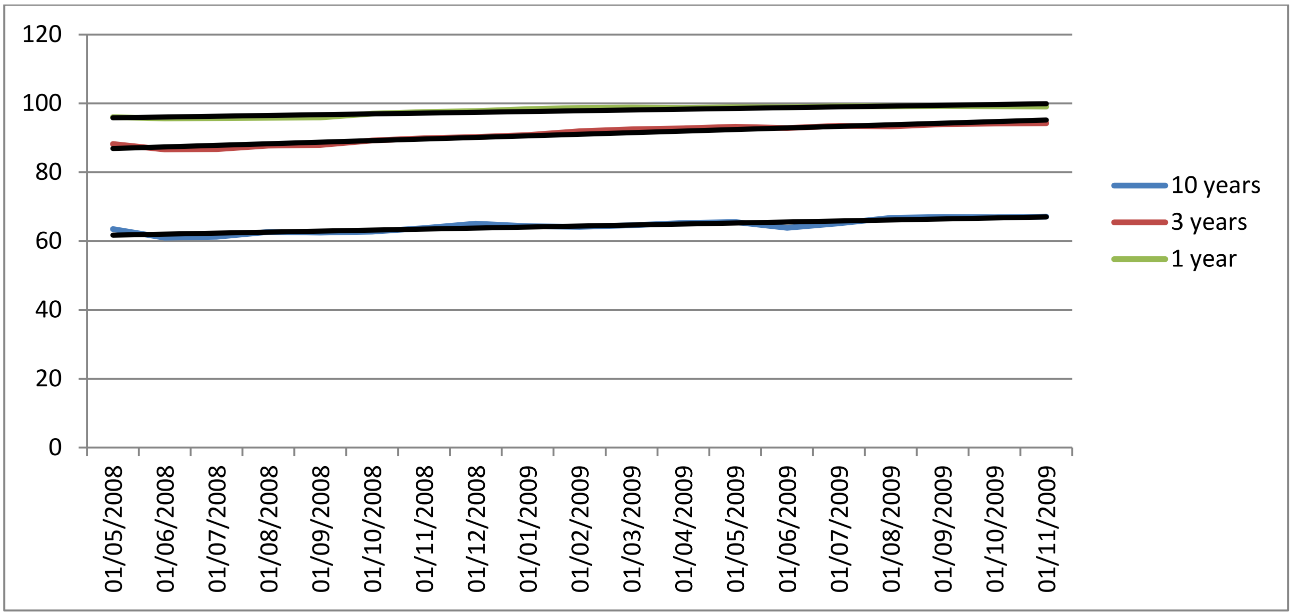

Figure 15. Positive trend of Sovereign Italian bonds prices for the period May 2008-November 2009.

For the opposite reason, the periods from September 2006 to December 2007 and from October 2010 to December 2011, the black lines representing a linear trend for each of the time series are upward sloping. Table 3 shows the percentage variation of each bond price on the four considered sub-periods.

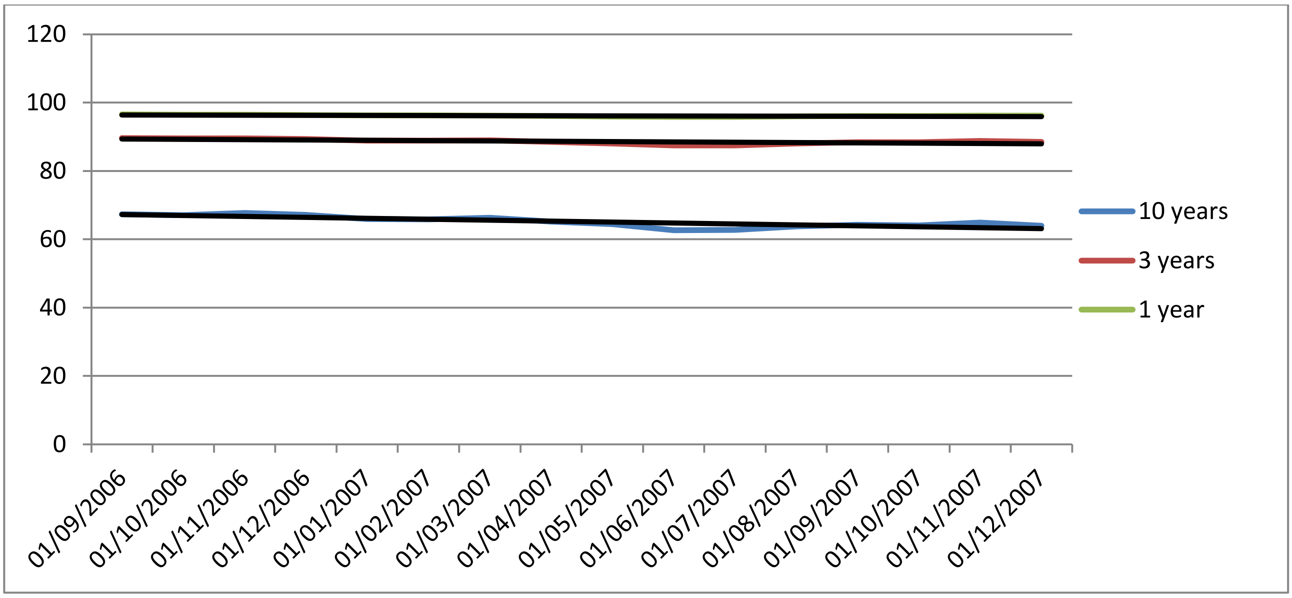

Figure 16. Negative trend of Sovereign Italian bonds prices for the period September 2006 - December 2007.

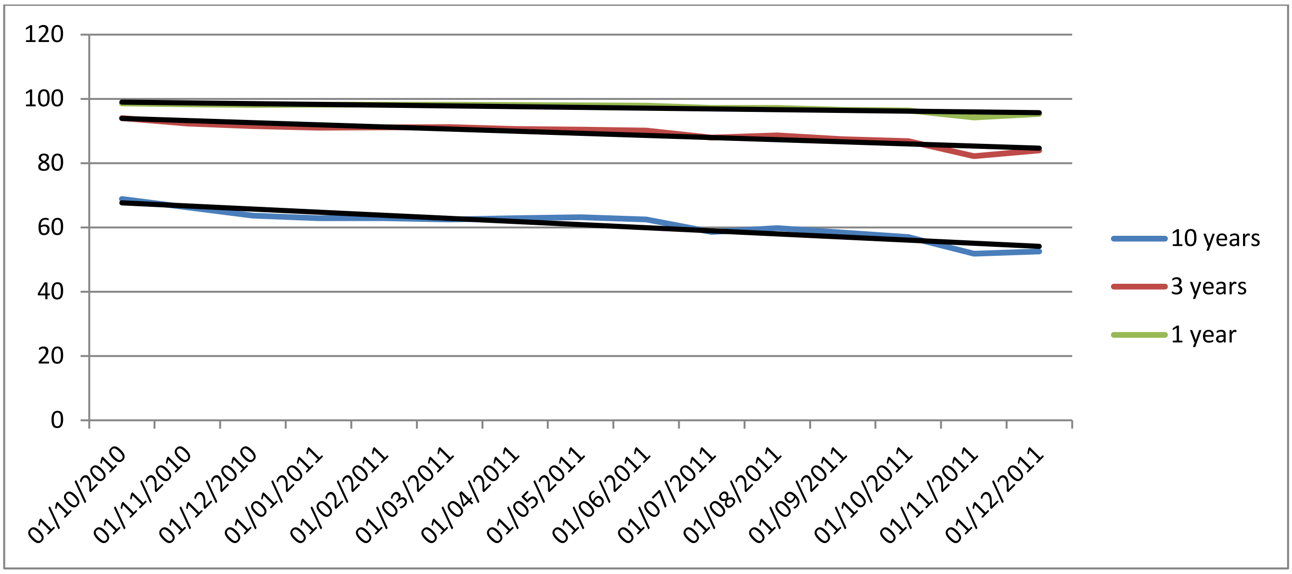

Figure 17. Negative trend of Sovereign Italian bonds prices for the period September 2006 - December 2007.

It can be noticed that the variations on bond prices are higher for the bonds with longer duration. The positive relationship risk-maturities simply derives from the price formulation of the bond:

$$P = C\left(\frac{1-(1+i)^{-N}}{i}\right) + \ M\ (1+i)^{-N}$$

The longer the maturity, the longer the number of payments that have to be actualized. So the wider left side of the equation simply force the right side to raise up. In table 3 we summarized the effect of these yield shocks.

T	10 years	3 years	1 year	Δ 10 years	Δ 3 years	Δ 1 year
Jan 2000	56,97942	86,63352	96,27604	23,81%	5,53%	1,11%
Jan 2006	70,5435	91,42094	97,3435			
Sep 2006	67,29713	89,65267	96,54373	-4,95%	-1,28%	-0,43%
Dec 2007	63,96301	88,50078	96,12889			
May 2008	63,48786	88,18038	95,99785	5,75%	6,90%	3,23%
Nov 2009	67,13564	94,26273	99,10017			
Oct 2010	68,80311	93,95838	98,5212	-23,69%	-10,70%	-3,27%
Dec 2011	52,50344	83,90255	95,298			

Table 3. Positive and negative price variation of Sovereign Italian bonds prices during falling and rising rate environments.

4.2 Vasicek calibration.

As we described in Chapter 2, the model introduced by Oldřich Vašíček in 1977 specifies the short term interest rate with follows the stochastic differential equation:

$$dr_t = a\,(b - r_t)dt + \sigma\, dz$$

The model is calibrated using least squares estimation. We consider the linear regression of Δr on r:

$$\Delta r = a(b - r)\Delta t + \sigma\varepsilon\sqrt{\Delta(t)}$$

The relationship between consecutive observations S_i and S_{i+1} is linear with a identically and normally distributed random term ε[46]:

$$S_{i+1} = aS_i + b + \varepsilon$$

The relationship between the linear fit and the model parameters is given by

$$a = e^{-\lambda\delta}$$

$$b = \mu\left(1 - e^{-\lambda\delta}\right)$$

$$sd(\varepsilon) = \sigma\sqrt{\frac{1 - e^{-2\lambda\delta}}{2\lambda}}$$

Where the term δ represents the model time step, that in our case is equal to 1 month (1/12). The above equations can be rewritten by isolating on the left side, respectively, the drift term μ, the speed of reversion λ, and the standard deviation σ:

$$\lambda = -\frac{\ln a}{\delta}$$

$$\mu = -\frac{b}{1 - a}$$

$$\sigma = sd(\varepsilon)\sqrt{\frac{-2\ln a}{\delta(1 - a^2)}}$$

The least square regression can be calculated using the following quantities:

- $S_x = \sum_{i=1}^{n} S_{i-1};$
- $S_y = \sum_{i=1}^{n} S_i;$

[46] Calibrating the Ornstein-Uhlenbeck (Vasicek) model. Source:http://www.sitmo.com

- $S_{xx} = \sum_{i=1}^{n} S_{i-1}^2$;
- $S_{xy} = \sum_{i=1}^{n} S_{i-1}\, S_i$;
- $S_{yy} = \sum_{i=1}^{n} S_i^2$

The parameters of the model can be now calculate with the following equations:

$$a = \frac{n\, S_{xy} - S_x S_y}{n\, S_{xx} - S_x^2}$$

$$b = \frac{S_y - a\, S_x}{n}$$

$$sd(\varepsilon) = \sqrt{\frac{n\, S_{yy} - S_x^2 - a(n\, S_{xy} - S_x S_y)}{n(n-2)}}$$

4.3 Cir calibration.

CIR model is a re-formulation of the Vasicek model. Indeed, the stochastic differential equation underlying this model is:

$$dr_t = a\,(b - r_t)dt + \sigma\sqrt{r}dz$$

Again, we are considering the least square calibration. We consider the linear regression of Δr on r:

$$\frac{r_{t+\Delta t} - r_t}{\sqrt{r_t}} = \frac{a\mu\, \Delta t}{\sqrt{r_t}} - a\sqrt{r_t}\Delta t + \sigma\varepsilon_t$$

The problem becomes[47]:

$$(a, \mu) = \arg\min \sum_{i=1}^{N-1} \left(\frac{r_{i+1} - r_{ti}}{\sqrt{r_{ti}}} - \frac{a\mu\, \Delta t}{\sqrt{r_{ti}}} + a\sqrt{r_{ti}}\Delta t \right)^2$$

The solutions to this problem are:

[47] K. Kladivko, "Maximum likelihood estimation of the Cox Ingersoll Ross process: the Matlab implementation." Working paper, Department of Statistics and Probability Calculus, University of Economics, Prague and Debt Management Department, Ministry of Finance of the Czech Republic, page 12.

$$a = \frac{N^2 - 2N + 1 + \sum_{i=1}^{N-1} r_{i+1} \sum_{i=1}^{N-1} \frac{1}{r_{ti}} - \sum_{i=1}^{N-1} r_{ti} \sum_{i=1}^{N-1} \frac{1}{r_{ti}} - (N-1) \sum_{i=1}^{N-1} \frac{r_{t_{i+1}}}{r_{ti}}}{\left(N^2 - 2N + 1 - \sum_{i=1}^{N-1} r_{i+1} \sum_{i=1}^{N-1} \frac{1}{r_{ti}}\right) \Delta t}$$

$$\mu = \frac{(N-1) + \sum_{i=1}^{N-1} r_{t_{i+1}} - \sum_{i=1}^{N-1} \frac{r_{t_{i+1}}}{r_{ti}} \sum_{i=1}^{N-1} r_{ti}}{N^2 - 2N + 1 + \sum_{i=1}^{N-1} r_{t_{i+1}} \sum_{i=1}^{N-1} \frac{1}{r_{ti}} - \sum_{i=1}^{N-1} r_{ti} \sum_{i=1}^{N-1} \frac{1}{r_{ti}} - (N-1) \sum_{i=1}^{N-1} \frac{r_{t_{i+1}}}{r_{ti}}}$$

The parameter σ is found as the standard deviation of residuals. Recalling the formulas described in Chapter two, we can now compute the entire shape of the term structure as a function of $r(t)$.

$$R(t,T) = -\frac{1}{T-t} \ln[A(t,T)] + \frac{1}{T-t} B(t,T) r(t)$$

Where:

- $A(t,T) = e^{\frac{[B(t,T)-T+t](a^2 b - \frac{\sigma^2}{2})}{a} - \frac{\sigma^2 B(t,T)^2}{4a}}$
- $B(t,T) = \frac{1-e^{-a(T-t)}}{a}$

are the terms of the Vasicek model, and

- $A(t,T) = \left\{\frac{2\gamma e^{(a+\gamma)(T-t)/2}}{(\gamma+a)[e^{\gamma(T-t)}-1]+2\gamma}\right\}^{2ab/\sigma^2}$
- $B(t,T) = \frac{2[e^{\gamma(T-t)}-1]}{(\gamma+a)[e^{\gamma(T-t)}-1]+2\gamma}$
- $\gamma = \sqrt{a^2 + 2\sigma^2}$

are the elements of CIR model.

4.4 Results of calibration process and interest rate projections.

The results of both calibrations and models projections are shown in Table 4.

The estimation were made by using Matlab software. Codes of Vasicek and CIR models are provided at the end of the chapter. The long term level μ turns to be higher in CIR simulation, while the speed of reversion is much faster looking at Vasicek calibration. The table shows also the terms A and B that allows the short rate forecasts and the estimation of zero rates at different maturities.

Vasicek	a	0,3083	Σ	1%	CIR	a	0,2	γ	0,21213
	b	0,03	-	-		b	0,041	Σ	5%
	T	10	3	1		T	10	3	1
	B(t, t+T)	3,09497	1,95728	0,86054		B(t, t+T)	4,25607	2,24968	0,906
	A(t,t+T)	0,80372	0,96732	0,99536		A(t,t+T)	0,79396	0,97	0,99617

Table 4. Result of Vasicek and CIR calibrations. The terms A and B are used to calculate the zero rates at 10, 3 and 1 year maturity.

Once the simulation has been run, we have a 12 months forward projections of yields level of the three bonds for the year 2012. Table 5 shows these results.

Vasick simulation			
t	10 years	3 years	1 year
1	0,0242	0,0161	0,0113
2	0,0241	0,0158	0,0109
3	0,0244	0,0165	0,0118
4	0,0224	0,0121	0,0061
5	0,0213	0,0099	0,0031
6	0,0219	0,0111	0,0047
7	0,0219	0,0112	0,0048
8	0,0218	0,0110	0,0045
9	0,0210	0,0093	0,0023
10	0,0207	0,0087	0,0015
11	0,0204	0,0080	0,0007
12	0,0207	0,0086	0,0014

CIR simulation			
t	10 years	3 years	1year
1	0,01048	0,01017	0,01006
2	0,01057	0,01028	0,01018
3	0,01052	0,01022	0,01011
4	0,01049	0,01018	0,01007
5	0,01054	0,01025	0,01015
6	0,01044	0,01012	0,01000
7	0,01039	0,01006	0,00994
8	0,01040	0,01006	0,00994
9	0,01037	0,01002	0,00990
10	0,01042	0,01009	0,00997
11	0,01044	0,01012	0,01001
12	0,01035	0,01000	0,00988

Table 5. Example of Vasicek and CIR interest rate estimations. The results describe 12 months forward projection for the yields of the three bonds.

It is important to note that the weakness of equilibrium models (Rendleman and Bartter, Vasicek, CIR) consists on the lack of perfect fit with the current term structure of interest rates (Figure 12 and Figure 13). Indeed, these models give the term structure as an output, based only on the current level of the short term rate. Equilibrium models can be transformed into no-arbitrage models (Chapter 2) which conversely, use the actual composition of the term structure as an input. Looking at the modeling part, the difference consists in the time depending of the drift term dt. No-arbitrage models consider the drift term as function of time and dependent on the short rate. They all aim to find adequate models for the movement of the actual interest-rate term structure. Following the works of Vasicek (1977) and of Cox et al. (1985), many other authors have implemented extensions which introduce multifactor extensions of the models like Beaglehole and

Tenney (1991)[48], Duffie and Kan (1996)[49], El Karoui et al. (1992)[50], and Jamshidian (1996)[51]. These extensions lead to generally better results in terms of fit to the data but they present a big disadvantages: computations on such are more complicated, and they generally face problems of over-parametrisation. Other authors proposed to a full maximum likelihood estimation[52].

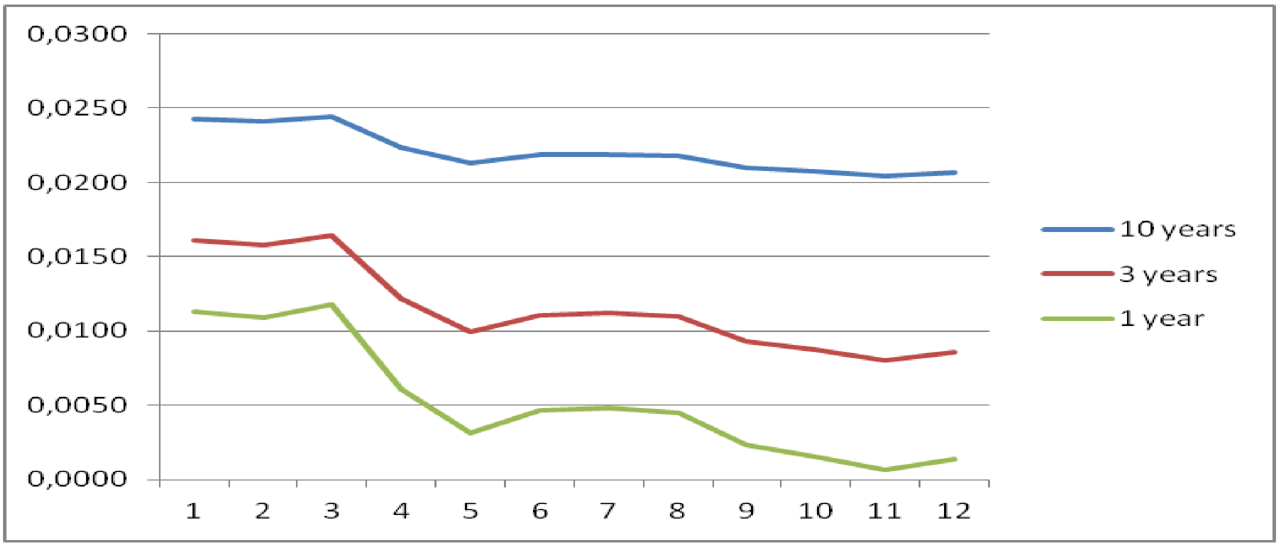

Figure 18. Vasicek model forecasts produce negative trend of interest rates for the year 2012.

[48] Beaglehole, D.R., Tenney, M.S., 1991, "General solution of some interest rate-contingent claim pricing equations.", The Journal of Fixed Income, 1, pp. 69-83-

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[52] K. Kladivko, "Maximum likelihood estimation of the Cox Ingersoll Ross process: Matlab implementation."

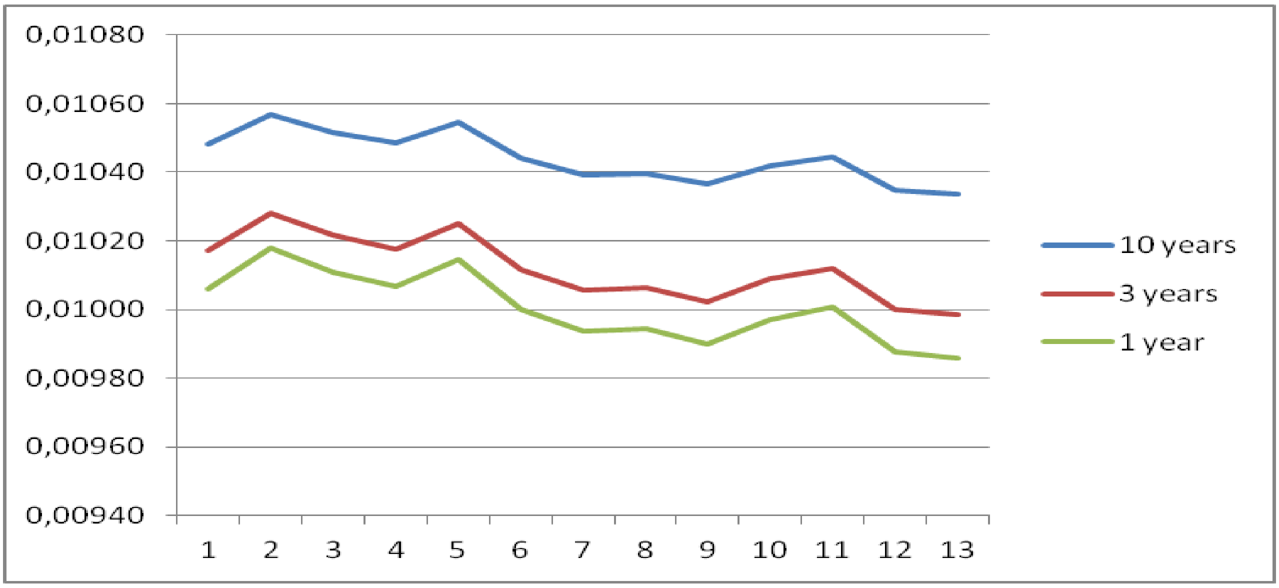

Figure 19. CIR model forecasts produce negative trend of interest rates for the year 2012.

In this work we are going to consider the variations given by 10 different independent simulations of interest rates and we apply the percentage variations on each yield to the real data. In such a way we can link the real interest rate levels to the two main important elements of the short rate models: the long term μ and the speed of reversion a towards that level.

4.5 Optimization model.

We saw in Chapter 3 that the goal of many investors is to replicate a certain benchmark return. In this case, the risk perception of an investor is not related to maturities mismatching or interest rates per se, but it is referred to the volatility of the difference between portfolio returns and benchmark returns.

The active management strategy applied on this work, consists in running an MADD model in order to outperforme a certain benchmark return. As we saw above, manager are concerned about negative deviations from the benchmark z^-, so the model is written in order to drop positive deviation z^+. Then additional constraint of non negative quotes have been added.

$$\min \sum_{t=1}^{T} z^-$$

Subject to

$$X_t\beta + z^- \geq Y_t$$
$$x_{i,t} \geq 0 \quad for\ i = 1,2,3\ ; t = 1, \dots, 10$$

We consider the realized yield on the real market and we apply to each bond's yield the 10 possible percentage variations derived from the CIR model simulations. This choice is due to the fact that Vasicek model tends to produce negative interest rate that are not applicable to the real data. The portfolio's value is given by:

$$X_t\beta = x_{1t}P_{1t} + x_{2t}P_{2t} + x_{3t}P_{2t} \qquad for\ t = 1, \dots, 10$$

The benchmark we refer to is an equally weighted portfolio composed by the three bonds. The benchmark's value is given by:

$$Y_t = \left(\frac{1}{3}\right)P_{1t} + \left(\frac{1}{3}\right)P_{2t} + \left(\frac{1}{3}\right)P_{3t} \quad for\ t = 1, \dots, 10$$

The initial capital is equal to 1.000 Euros. Results are shown in The definition given to the benchmark implies that for each *t* the weights are the same in terms of quote's value, while the quote values given by the optimized portfolio change over the time depending on the optimal weights.

Once the optimal composition has been calculated we can apply the real interest rate to each quote values and find the new capital at the end of each period:

$$New\ capital = x_{1t}P_{1t} * (1 + r_{1t})^{1/12} + x_{2t}P_{2t} * (1 + r_{2t})^{1/12} + x_{3t}P_{2t} * (1 + r_{3t})^{1/12}$$

Where r_{it} represent the annualized observable interest rate on the market at time *t*.

time		Benchmark weights			Portfolio weights		
t1	Quotes	6,2889	3,8661	3,4353	5,8054	4,4672	3,1659
	Values	333,3333	333,3333	333,3333	307,7032	385,1578	307,1863
t2	Quotes	5,7181	3,6923	3,4025	4,5665	5,4404	2,4769
	Values	333,3333	333,3333	333,3333	266,1983	491,1475	242,6542
t3	Quotes	5,4126	3,6258	3,3808	4,5317	5,1594	2,5011
	Values	333,3333	333,3333	333,3333	279,0832	474,3212	246,5956
t4	Quotes	5,7030	3,7280	3,4100	2,8890	4,3163	4,5543
	Values	333,3333	333,3333	333,3333	168,8613	385,9393	445,1994
t5	Quotes	5,7797	3,7560	3,4195	4,6097	5,4896	2,5335
	Values	333,3333	333,3333	333,3333	265,8559	487,1809	246,9632
t6	Quotes	5,9263	3,8406	3,4503	2,9064	4,3965	4,7091
	Values	333,3333	333,3333	333,3333	163,4730	381,5781	454,9489
t7	Quotes	5,9921	3,8232	3,4325	5,1102	5,9183	2,0568
	Values	333,3333	333,3333	333,3333	284,2738	515,9880	199,7382
t8	Quotes	5,8833	3,7457	3,4149	5,1149	5,6139	2,1576
	Values	333,3333	333,33333	333,3333	289,7967	499,5952	210,6081
t9	Quotes	5,5151	3,6384	3,3919	5,2099	5,2492	2,0778
	Values	333,3333	333,33333	333,3333	314,8924	480,9112	204,1964
t10	Quotes	5,4090	3,6243	3,3905	5,2108	5,2421	2,0013
	Values	333,3333	333,33333	333,3333	321,1199	482,1258	196,7543
t11	Quotes	5,3510	3,6064	3,3856	4,7374	4,9836	2,4809
	Values	333,3333	333,33333	333,3333	295,1072	460,6322	244,2605
t12	Quotes	5,1900	3,5808	3,3797	5,5000	4,2508	4,0278
	Values	333,3333	333,33333	333,3333	207,0515	395,6976	397,2509

Table 5. Example of Vasicek and CIR interest rate estimations. The results describe 12 months forward projection for the yields of the three bonds.

Figure 14 shows the new capital obtained by investing on the optimized portfolio against the new capital obtained by investing on the benchmark, calculated using the formula above. As we can see, the portfolio values are higher on times 1,2,3,5,7,8,9,10,11 while the benchmark performs better only in times 4,6 and 12. This results can be attributed to the sensibility on yield variations of the optimized portfolio. Rather than equally divided the proportions of the capital on the three classes, the optimized portfolio moves the quotes distribution depending on the future values of interest rate, and in turn of the prices. The same principle has been applied to other possible strategies that an active manager can take. Figure 15 shows a barbell strategy where the portfolio's composition is concentrated on 10 years and 1 year bond. The performance in this case is lower than the equally weighted strategy imposed by the benchmark. A third comparison has been made between the benchmark and a bullet strategy where the portfolio's composition is concentrated on 1 year bond. Figure 16 shows that this strategy obtains a lower performance with respect to the benchmark during all the considered periods.

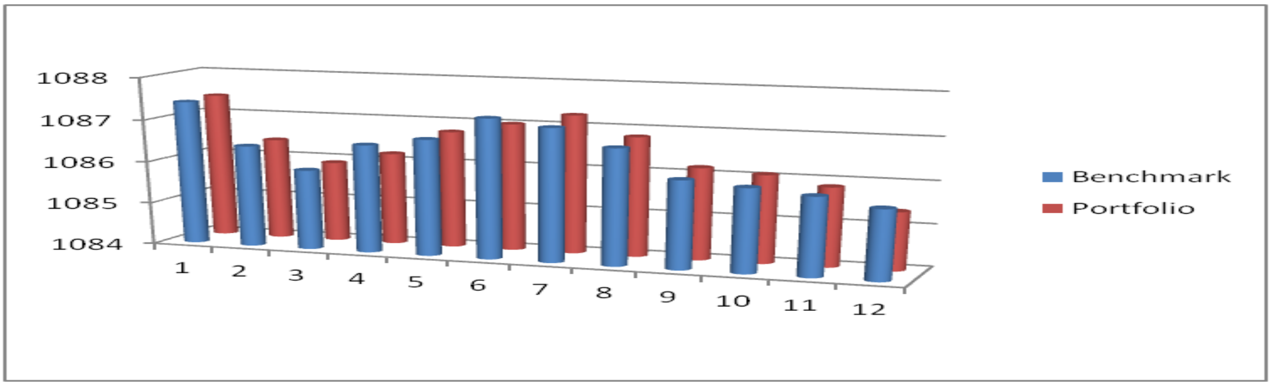

Figure 20. Performance of optimize portfolio against the benchmark for the year 2012.

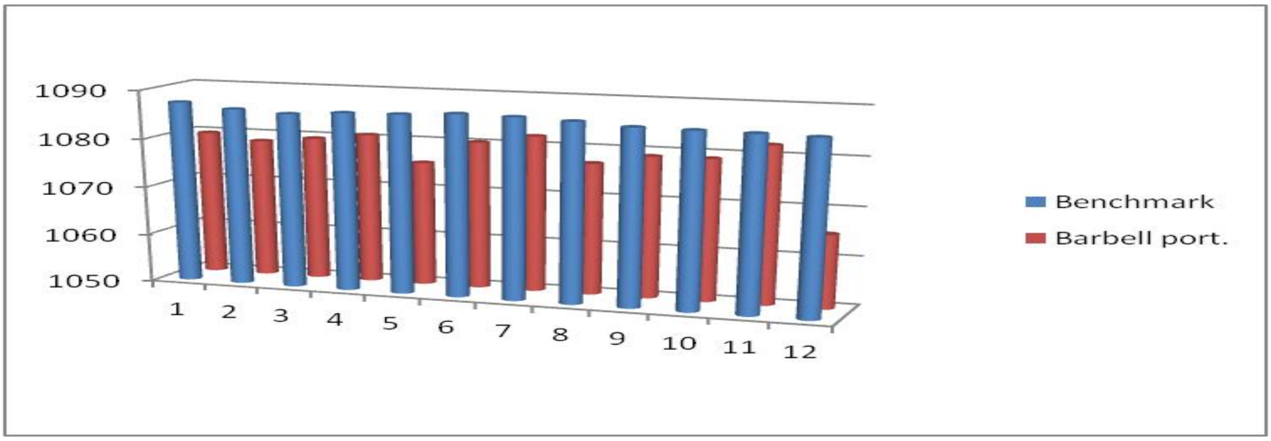

Figure 21. Performance of Barbell portfolio against the benchmark for the year 2012.

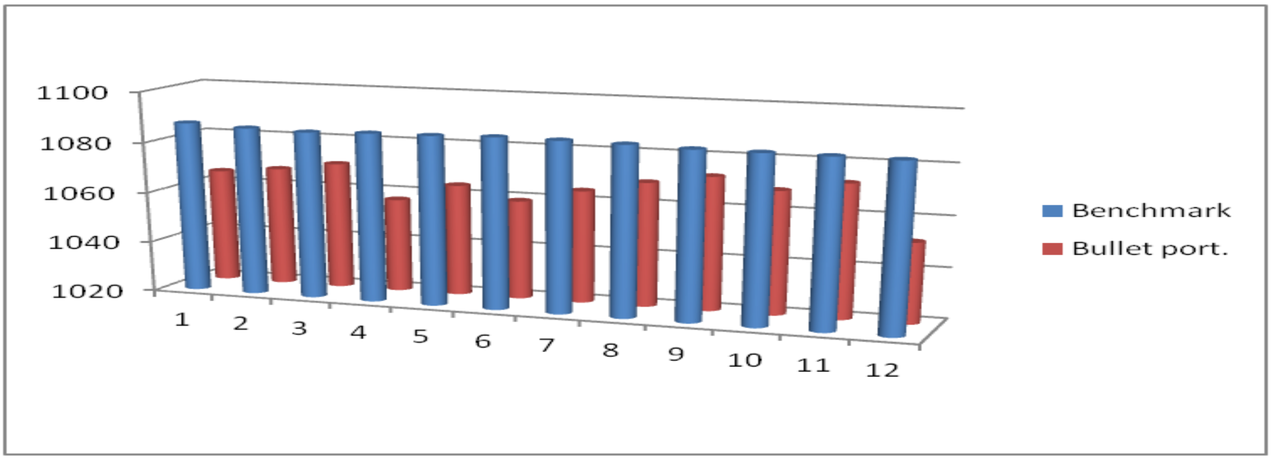

Figure 22. Performance of bullet portfolio against the benchmark for the year 2012.

Conclusions

In this work we have discussed how important are the analysis of interest rate movements for an investor who chooses to operate with fixed income securities. All the possible manager strategies are somehow related to interest rate trends and this is why it is important to consider this element during a portfolio selection process. We demonstrated how it is possible to calibrate a short term model and how we can exploit the results to create expectations about the future term structure's trend. We have shown that optimized movements of portfolio's weights generates higher return for the investor who undertakes an active strategy whit respect to a benchmark portfolio who maintains the same portion of capital in each asset class.

APPENDIX

Least Squares Calibration Vasicek model (source: "Calibrating the Ornstein-Uhlenbeck (Vasicek) model", http://www.sitmo.com)

```
function [mu,sigma,lambda] = OU_Calibrate_LS(S,delta)
  n = length(S)-1;

  Sx  = sum( S(1:end-1) );
  Sy  = sum( S(2:end) );
  Sxx = sum( S(1:end-1).^2 );
  Sxy = sum( S(1:end-1).*S(2:end) );
  Syy = sum( S(2:end).^2 );

  a  = ( n*Sxy - Sx*Sy ) / ( n*Sxx -Sx^2 );
  b  = ( Sy - a*Sx ) / n;
  sd = sqrt( (n*Syy - Sy^2 - a*(n*Sxy - Sx*Sy) )/n/(n-2) );

  lambda = -log(a)/delta;
  mu     = b/(1-a);
  sigma  =  sd * sqrt( -2*log(a)/delta/(1-a^2) );
end
```

Least Squares Calibration CIR model (source "Maximum likelihood estimation of the Cox Ingersoll Ross process: Matlab implementation." Working paper, Department of Statistics and Probability Calculus, University of Economics, Prague and Debt Management Department, Ministry of Finance of the Czech Republic.)

```
x = Model.Data(1:end-1); % Time series of interest rates observations
dx = diff(Model.Data);
dx = dx./x.^0.5;
regressors = [Model.TimeStep./x.^0.5, Model.TimeStep*x.^0.5];
drift = regressors\dx; % OLS regressors coefficients estimates
res = regressors*drift - dx;
alpha = -drift(2);
mu = -drift(1)/drift(2);
sigma = sqrt(var(res, 1)/Model.TimeStep);
InitialParams = [alpha mu sigma];
```

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